Chapter Two

Vector Spaces

The first chapter finished with a fair understanding of how Gauss's Method solves a linear system. It systematically takes linear combinations of the rows. Here we move to a general study of linear combinations.

We need a setting. At times in the first chapter we've combined vectors from $\mathbb{R}^2$, at other times vectors from $\mathbb{R}^3$, and at other times vectors from higher-dimensional spaces. So our first impulse might be to work in $\mathbb{R}^n$, leaving n unspecified. This would have the advantage that any of the results would hold for $\mathbb{R}^2$ and for $\mathbb{R}^3$ and for many other spaces, simultaneously.

But if having the results apply to many spaces at once is advantageous then sticking only to $\mathbb{R}^n$'s is overly restrictive. We'd like our results to apply to combinations of row vectors, as in the final section of the first chapter. We've even seen some spaces that are not simply a collection of all of the same-sized column vectors or row vectors. For instance, we've seen a homogeneous system's solution set that is a plane inside of $\mathbb{R}^3$. This set is a closed system in that a linear combination of these solutions is also a solution. But it does not contain all of the three-tall column vectors, only some of them.

We want the results about linear combinations to apply anywhere that linear combinations make sense. We shall call any such set a *vector space*. Our results, instead of being phrased as "Whenever we have a collection in which we can sensibly take linear combinations …", will be stated "In any vector space …"

Such a statement describes at once what happens in many spaces. To understand the advantages of moving from studying a single space to studying a class of spaces, consider this analogy. Imagine that the government made laws one person at a time: "Leslie Jones can't jay walk." That would be bad; statements have the virtue of economy when they apply to many cases at once. Or suppose that they said, "Kim Ke must stop when passing an accident." Contrast that with, "Any doctor must stop when passing an accident." More general statements, in some ways, are clearer.

I Definition of Vector Space

We shall study structures with two operations, an addition and a scalar multiplication, that are subject to some simple conditions. We will reflect more on the conditions later but on first reading notice how reasonable they are. For instance, surely any operation that can be called an addition (e.g., column vector addition, row vector addition, or real number addition) will satisfy conditions (1) through (5) below.

I.1 Definition and Examples

1.1 Definition A *vector space* (over $\mathbb{R}$) consists of a set V along with two operations '+' and '·' subject to the conditions that for all vectors $\vec{v}, \vec{w}, \vec{u} \in V$ and all *scalars* $r, s \in \mathbb{R}$:

(1) the set V is closed under vector addition, that is, $\vec{v} + \vec{w} \in V$
(2) vector addition is commutative, $\vec{v} + \vec{w} = \vec{w} + \vec{v}$
(3) vector addition is associative, $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$
(4) there is a *zero vector* $\vec{0} \in V$ such that $\vec{v} + \vec{0} = \vec{v}$ for all $\vec{v} \in V$
(5) each $\vec{v} \in V$ has an *additive inverse* $\vec{w} \in V$ such that $\vec{w} + \vec{v} = \vec{0}$
(6) the set V is closed under scalar multiplication, that is, $r \cdot \vec{v} \in V$
(7) addition of scalars distributes over scalar multiplication, $(r+s)\cdot\vec{v} = r\cdot\vec{v}+s\cdot\vec{v}$
(8) scalar multiplication distributes over vector addition, $r\cdot(\vec{v}+\vec{w}) = r\cdot\vec{v}+r\cdot\vec{w}$
(9) ordinary multipication of scalars associates with scalar multiplication, $(rs) \cdot \vec{v} = r \cdot (s \cdot \vec{v})$
(10) multiplication by the scalar 1 is the identity operation, $1 \cdot \vec{v} = \vec{v}$.

1.2 Remark The definition involves two kinds of addition and two kinds of multiplication, and so may at first seem confused. For instance, in condition (7) the '+' on the left is addition of two real numbers while the '+' on the right is addition of two vectors in V. These expressions aren't ambiguous because of context; for example, r and s are real numbers so '$r + s$' can only mean real number addition. In the same way, item (9)'s left side 'rs' is ordinary real number multiplication, while its right side '$s \cdot \vec{v}$' is the scalar multipliction defined for this vector space.

The best way to understand the definition is to go through the examples below and for each, check all ten conditions. The first example includes that check, written out at length. Use it as a model for the others. Especially important are the *closure* conditions, (1) and (6). They specify that the addition and scalar multiplication operations are always sensible — they are defined for every pair of vectors and every scalar and vector, and the result of the operation is a member of the set (see Example 1.4).

1.3 Example The set $\mathbb{R}^2$ is a vector space if the operations '+' and '·' have their usual meaning.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} \qquad r \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} rx_1 \\ rx_2 \end{pmatrix}$$

We shall check all of the conditions.

There are five conditions in the paragraph having to do with addition. For (1), closure of addition, observe that for any $v_1, v_2, w_1, w_2 \in \mathbb{R}$ the result of the vector sum

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix}$$

is a column array with two real entries, and so is in $\mathbb{R}^2$. For (2), that addition of vectors commutes, take all entries to be real numbers and compute

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix} = \begin{pmatrix} w_1 + v_1 \\ w_2 + v_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

(the second equality follows from the fact that the components of the vectors are real numbers, and the addition of real numbers is commutative). Condition (3), associativity of vector addition, is similar.

$$\begin{aligned} (\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}) + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \begin{pmatrix} (v_1 + w_1) + u_1 \\ (v_2 + w_2) + u_2 \end{pmatrix} \\ &= \begin{pmatrix} v_1 + (w_1 + u_1) \\ v_2 + (w_2 + u_2) \end{pmatrix} \\ &= \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + (\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}) \end{aligned}$$

For the fourth condition we must produce a zero element — the vector of zeroes is it.

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

For (5), to produce an additive inverse, note that for any $v_1, v_2 \in \mathbb{R}$ we have

$$\begin{pmatrix} -v_1 \\ -v_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so the first vector is the desired additive inverse of the second.

The checks for the five conditions having to do with scalar multiplication are similar. For (6), closure under scalar multiplication, where $r, v_1, v_2 \in \mathbb{R}$,

$$r \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} rv_1 \\ rv_2 \end{pmatrix}$$

is a column array with two real entries, and so is in $\mathbb{R}^2$. Next, this checks (7).

$$(r+s) \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} (r+s)v_1 \\ (r+s)v_2 \end{pmatrix} = \begin{pmatrix} rv_1 + sv_1 \\ rv_2 + sv_2 \end{pmatrix} = r \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + s \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

For (8), that scalar multiplication distributes from the left over vector addition, we have this.

$$r \cdot (\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}) = \begin{pmatrix} r(v_1 + w_1) \\ r(v_2 + w_2) \end{pmatrix} = \begin{pmatrix} rv_1 + rw_1 \\ rv_2 + rw_2 \end{pmatrix} = r \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + r \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

The ninth

$$(rs) \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} (rs)v_1 \\ (rs)v_2 \end{pmatrix} = \begin{pmatrix} r(sv_1) \\ r(sv_2) \end{pmatrix} = r \cdot (s \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix})$$

and tenth conditions are also straightforward.

$$1 \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1v_1 \\ 1v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

In a similar way, each $\mathbb{R}^n$ is a vector space with the usual operations of vector addition and scalar multiplication. (In $\mathbb{R}^1$, we usually do not write the members as column vectors, i.e., we usually do not write '(π)'. Instead we just write 'π'.)

1.4 Example This subset of $\mathbb{R}^3$ that is a plane through the origin

$$P = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x + y + z = 0 \}$$

is a vector space if '+' and '$\cdot$' are interpreted in this way.

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} \qquad r \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rx \\ ry \\ rz \end{pmatrix}$$

The addition and scalar multiplication operations here are just the ones of $\mathbb{R}^3$, reused on its subset P. We say that P *inherits* these operations from $\mathbb{R}^3$. This example of an addition in P

$$\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

illustrates that P is closed under addition. We've added two vectors from P — that is, with the property that the sum of their three entries is zero — and the result is a vector also in P. Of course, this example is not a proof. For the proof that P is closed under addition, take two elements of P.

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \quad \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

Membership in P means that $x_1 + y_1 + z_1 = 0$ and $x_2 + y_2 + z_2 = 0$. Observe that their sum

$$\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$$

is also in P since its entries add $(x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) = (x_1 + y_1 + z_1) + (x_2 + y_2 + z_2)$ to 0. To show that P is closed under scalar multiplication, start with a vector from P

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

where $x + y + z = 0$, and then for $r \in \mathbb{R}$ observe that the scalar multiple

$$r \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rx \\ ry \\ rz \end{pmatrix}$$

gives $rx + ry + rz = r(x + y + z) = 0$. Thus the two closure conditions are satisfied. Verification of the other conditions in the definition of a vector space are just as straightforward.

1.5 Example Example 1.3 shows that the set of all two-tall vectors with real entries is a vector space. Example 1.4 gives a subset of an $\mathbb{R}^n$ that is also a vector space. In contrast with those two, consider the set of two-tall columns with entries that are integers (under the usual operations of component-wise addition and scalar multiplication). This is a subset of a vector space but it is not itself a vector space. The reason is that this set is not closed under scalar multiplication, that is, it does not satisfy condition (6). Here is a column with integer entries and a scalar such that the outcome of the operation

$$0.5 \cdot \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1.5 \end{pmatrix}$$

is not a member of the set, since its entries are not all integers.

1.6 Example The singleton set

$$\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \}$$

is a vector space under the operations

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad r \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

that it inherits from $\mathbb{R}^4$.

A vector space must have at least one element, its zero vector. Thus a one-element vector space is the smallest possible.

1.7 Definition A one-element vector space is a *trivial* space.

The examples so far involve sets of column vectors with the usual operations. But vector spaces need not be collections of column vectors, or even of row vectors. Below are some other types of vector spaces. The term 'vector space' does not mean 'collection of columns of reals'. It means something more like 'collection in which any linear combination is sensible'.

1.8 Example Consider $\mathcal{P}_3 = \{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_0, \ldots, a_3 \in \mathbb{R}\}$, the set of polynomials of degree three or less (in this book, we'll take constant polynomials, including the zero polynomial, to be of degree zero). It is a vector space under the operations

$$\begin{aligned} &(a_0 + a_1x + a_2x^2 + a_3x^3) + (b_0 + b_1x + b_2x^2 + b_3x^3) \\ &\qquad = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3 \end{aligned}$$

and

$$r \cdot (a_0 + a_1x + a_2x^2 + a_3x^3) = (ra_0) + (ra_1)x + (ra_2)x^2 + (ra_3)x^3$$

(the verification is easy). This vector space is worthy of attention because these are the polynomial operations familiar from high school algebra. For instance, $3 \cdot (1 - 2x + 3x^2 - 4x^3) - 2 \cdot (2 - 3x + x^2 - (1/2)x^3) = -1 + 7x^2 - 11x^3$.

Although this space is not a subset of any $\mathbb{R}^n$, there is a sense in which we can think of $\mathcal{P}_3$ as "the same" as $\mathbb{R}^4$. If we identify these two space's elements in this way

$$a_0 + a_1x + a_2x^2 + a_3x^3 \quad \text{corresponds to} \quad \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

then the operations also correspond. Here is an example of corresponding additions.

$$\begin{array}{rr} & 1 - 2x + 0x^2 + 1x^3 \\ + & 2 + 3x + 7x^2 - 4x^3 \\ \hline & 3 + 1x + 7x^2 - 3x^3 \end{array} \quad \text{corresponds to} \quad \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ 7 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 7 \\ -3 \end{pmatrix}$$

Things we are thinking of as "the same" add to "the same" sum. Chapter Three makes precise this idea of vector space correspondence. For now we shall just leave it as an intuition.

1.9 Example The set $\mathcal{M}_{2\times2}$ of 2×2 matrices with real number entries is a vector space under the natural entry-by-entry operations.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} a+w & b+x \\ c+y & d+z \end{pmatrix} \qquad r \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix}$$

As in the prior example, we can think of this space as "the same" as $\mathbb{R}^4$.

1.10 Example The set $\{f \mid f\colon \mathbb{N} \to \mathbb{R}\}$ of all real-valued functions of one natural number variable is a vector space under the operations

$$(f_1 + f_2)\,(n) = f_1(n) + f_2(n) \qquad (r \cdot f)\,(n) = r\,f(n)$$

so that if, for example, $f_1(n) = n^2 + 2\sin(n)$ and $f_2(n) = -\sin(n) + 0.5$ then $(f_1 + 2f_2)\,(n) = n^2 + 1$.

We can view this space as a generalization of Example 1.3 — instead of 2-tall vectors, these functions are like infinitely-tall vectors.

$$\begin{array}{c|c} n & f(n) = n^2 + 1 \\ \hline 0 & 1 \\ 1 & 2 \\ 2 & 5 \\ 3 & 10 \\ \vdots & \vdots \end{array} \quad \text{corresponds to} \quad \begin{pmatrix} 1 \\ 2 \\ 5 \\ 10 \\ \vdots \end{pmatrix}$$

Addition and scalar multiplication are component-wise, as in Example 1.3. (We can formalize "infinitely-tall" by saying that it means an infinite sequence, or that it means a function from $\mathbb{N}$ to $\mathbb{R}$.)

1.11 Example The set of polynomials with real coefficients

$$\{a_0 + a_1x + \cdots + a_nx^n \mid n \in \mathbb{N} \text{ and } a_0, \ldots, a_n \in \mathbb{R}\}$$

makes a vector space when given the natural '+'

$$\begin{aligned}(a_0 + a_1x + \cdots + a_nx^n) &+ (b_0 + b_1x + \cdots + b_nx^n) \\ &= (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n\end{aligned}$$

and '$\cdot$'.

$$r \cdot (a_0 + a_1x + \ldots a_nx^n) = (ra_0) + (ra_1)x + \ldots (ra_n)x^n$$

This space differs from the space $\mathcal{P}_3$ of Example 1.8. This space contains not just degree three polynomials, but degree thirty polynomials and degree three hundred polynomials, too. Each individual polynomial of course is of a finite degree, but the set has no single bound on the degree of all of its members.

We can think of this example, like the prior one, in terms of infinite-tuples. For instance, we can think of $1 + 3x + 5x^2$ as corresponding to $(1, 3, 5, 0, 0, \ldots)$. However, this space differs from the one in Example 1.10. Here, each member of the set has a finite degree, that is, under the correspondence there is no element from this space matching $(1, 2, 5, 10, \ \ldots\ )$. Vectors in this space correspond to infinite-tuples that end in zeroes.

1.12 Example The set $\{f \mid f\colon \mathbb{R} \to \mathbb{R}\}$ of all real-valued functions of one real variable is a vector space under these.

$$(f_1 + f_2)\,(x) = f_1(x) + f_2(x) \qquad (r \cdot f)\,(x) = r\,f(x)$$

The difference between this and Example 1.10 is the domain of the functions.

1.13 Example The set $F = \{a\cos\theta + b\sin\theta \mid a, b \in \mathbb{R}\}$ of real-valued functions of the real variable θ is a vector space under the operations

$$(a_1\cos\theta + b_1\sin\theta) + (a_2\cos\theta + b_2\sin\theta) = (a_1 + a_2)\cos\theta + (b_1 + b_2)\sin\theta$$

and

$$r \cdot (a\cos\theta + b\sin\theta) = (ra)\cos\theta + (rb)\sin\theta$$

inherited from the space in the prior example. (We can think of F as "the same" as $\mathbb{R}^2$ in that $a\cos\theta + b\sin\theta$ corresponds to the vector with components a and b.)

1.14 Example The set

$$\{f\colon \mathbb{R}\to\mathbb{R} \mid \frac{d^2f}{dx^2}+f=0\}$$

is a vector space under the, by now natural, interpretation.

$$(f+g)\,(x)=f(x)+g(x) \qquad (r\cdot f)\,(x)=r\,f(x)$$

In particular, notice that closure is a consequence

$$\frac{d^2(f+g)}{dx^2}+(f+g)=(\frac{d^2f}{dx^2}+f)+(\frac{d^2g}{dx^2}+g)$$

and

$$\frac{d^2(rf)}{dx^2}+(rf)=r(\frac{d^2f}{dx^2}+f)$$

of basic Calculus. This turns out to equal the space from the prior example— functions satisfying this differential equation have the form $a\cos\theta+b\sin\theta$— but this description suggests an extension to solutions sets of other differential equations.

1.15 Example The set of solutions of a homogeneous linear system in n variables is a vector space under the operations inherited from $\mathbb{R}^n$. For example, for closure under addition consider a typical equation in that system $c_1x_1+\cdots+c_nx_n=0$ and suppose that both these vectors

$$\vec{v}=\begin{pmatrix}v_1\\ \vdots\\ v_n\end{pmatrix} \qquad \vec{w}=\begin{pmatrix}w_1\\ \vdots\\ w_n\end{pmatrix}$$

satisfy the equation. Then their sum $\vec{v}+\vec{w}$ also satisfies that equation: $c_1(v_1+w_1)+\cdots+c_n(v_n+w_n)=(c_1v_1+\cdots+c_nv_n)+(c_1w_1+\cdots+c_nw_n)=0$. The checks of the other vector space conditions are just as routine.

We often omit the multiplication symbol '$\cdot$' between the scalar and the vector. We distinguish the multiplication in c_1v_1 from that in $r\vec{v}$ by context, since if both multiplicands are real numbers then it must be real-real multiplication while if one is a vector then it must be scalar-vector multiplication.

Example 1.15 has brought us full circle since it is one of our motivating examples. Now, with some feel for the kinds of structures that satisfy the definition of a vector space, we can reflect on that definition. For example, why specify in the definition the condition that $1\cdot\vec{v}=\vec{v}$ but not a condition that $0\cdot\vec{v}=\vec{0}$?

One answer is that this is just a definition— it gives the rules and you need to follow those rules to continue.

Another answer is perhaps more satisfying. People in this area have worked to develop the right balance of power and generality. This definition is shaped so that it contains the conditions needed to prove all of the interesting and important properties of spaces of linear combinations. As we proceed, we shall derive all of the properties natural to collections of linear combinations from the conditions given in the definition.

The next result is an example. We do not need to include these properties in the definition of vector space because they follow from the properties already listed there.

1.16 Lemma In any vector space V, for any $\vec{v} \in V$ and $r \in \mathbb{R}$, we have (1) $0 \cdot \vec{v} = \vec{0}$, (2) $(-1 \cdot \vec{v}) + \vec{v} = \vec{0}$, and (3) $r \cdot \vec{0} = \vec{0}$.

Proof For (1) note that $\vec{v} = (1+0) \cdot \vec{v} = \vec{v} + (0 \cdot \vec{v})$. Add to both sides the additive inverse of $\vec{v}$, the vector $\vec{w}$ such that $\vec{w} + \vec{v} = \vec{0}$.

$$\begin{aligned} \vec{w} + \vec{v} &= \vec{w} + \vec{v} + 0 \cdot \vec{v} \\ \vec{0} &= \vec{0} + 0 \cdot \vec{v} \\ \vec{0} &= 0 \cdot \vec{v} \end{aligned}$$

Item (2) is easy: $(-1 \cdot \vec{v}) + \vec{v} = (-1+1) \cdot \vec{v} = 0 \cdot \vec{v} = \vec{0}$. For (3), $r \cdot \vec{0} = r \cdot (0 \cdot \vec{0}) = (r \cdot 0) \cdot \vec{0} = \vec{0}$ will do. QED

The second item shows that we can write the additive inverse of $\vec{v}$ as '$-\vec{v}$' without worrying about any confusion with $(-1) \cdot \vec{v}$.

A recap: our study in Chapter One of Gaussian reduction led us to consider collections of linear combinations. So in this chapter we have defined a vector space to be a structure in which we can form such combinations, subject to simple conditions on the addition and scalar multiplication operations. In a phrase: vector spaces are the right context in which to study linearity.

From the fact that it forms a whole chapter, and especially because that chapter is the first one, a reader could suppose that our purpose in this book is the study of linear systems. The truth is that we will not so much use vector spaces in the study of linear systems as we instead have linear systems start us on the study of vector spaces. The wide variety of examples from this subsection shows that the study of vector spaces is interesting and important in its own right. Linear systems won't go away. But from now on our primary objects of study will be vector spaces.

Exercises

1.17 Name the zero vector for each of these vector spaces.

(a) The space of degree three polynomials under the natural operations.

(b) The space of 2×4 matrices.
(c) The space $\{f\colon [0..1] \to \mathbb{R} \mid f \text{ is continuous}\}$.
(d) The space of real-valued functions of one natural number variable.

✓ 1.18 Find the additive inverse, in the vector space, of the vector.
(a) In $\mathcal{P}_3$, the vector $-3-2x+x^2$.
(b) In the space 2×2,

$$\begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}.$$

(c) In $\{ae^x + be^{-x} \mid a, b \in \mathbb{R}\}$, the space of functions of the real variable x under the natural operations, the vector $3e^x - 2e^{-x}$.

✓ 1.19 For each, list three elements and then show it is a vector space.
(a) The set of linear polynomials $\mathcal{P}_1 = \{a_0 + a_1 x \mid a_0, a_1 \in \mathbb{R}\}$ under the usual polynomial addition and scalar multiplication operations.
(b) The set of linear polynomials $\{a_0 + a_1 x \mid a_0 - 2a_1 = 0\}$, under the usual polynomial addition and scalar multiplication operations.

Hint. Use Example 1.3 as a guide. Most of the ten conditions are just verifications.

1.20 For each, list three elements and then show it is a vector space.
(a) The set of 2×2 matrices with real entries under the usual matrix operations.
(b) The set of 2×2 matrices with real entries where the $2, 1$ entry is zero, under the usual matrix operations.

✓ 1.21 For each, list three elements and then show it is a vector space.
(a) The set of three-component row vectors with their usual operations.
(b) The set

$$\{\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \mid x+y-z+w=0\}$$

under the operations inherited from $\mathbb{R}^4$.

✓ 1.22 Show that each of these is not a vector space. (*Hint.* Check closure by listing two members of each set and trying some operations on them.)
(a) Under the operations inherited from $\mathbb{R}^3$, this set

$$\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid x+y+z=1\}$$

(b) Under the operations inherited from $\mathbb{R}^3$, this set

$$\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\}$$

(c) Under the usual matrix operations,

$$\{\begin{pmatrix} a & 1 \\ b & c \end{pmatrix} \mid a, b, c \in \mathbb{R}\}$$

(d) Under the usual polynomial operations,

$$\{a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R}^+\}$$

where $\mathbb{R}^+$ is the set of reals greater than zero

(e) Under the inherited operations,

$$\{\begin{pmatrix}x\\y\end{pmatrix}\in\mathbb{R}^2 \mid x+3y=4 \text{ and } 2x-y=3 \text{ and } 6x+4y=10\}$$

1.23 Define addition and scalar multiplication operations to make the complex numbers a vector space over $\mathbb{R}$.

✓ 1.24 Is the set of rational numbers a vector space over $\mathbb{R}$ under the usual addition and scalar multiplication operations?

1.25 Show that the set of linear combinations of the variables x, y, z is a vector space under the natural addition and scalar multiplication operations.

1.26 Prove that this is not a vector space: the set of two-tall column vectors with real entries subject to these operations.

$$\begin{pmatrix}x_1\\y_1\end{pmatrix}+\begin{pmatrix}x_2\\y_2\end{pmatrix}=\begin{pmatrix}x_1-x_2\\y_1-y_2\end{pmatrix} \qquad r\cdot\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}rx\\ry\end{pmatrix}$$

1.27 Prove or disprove that $\mathbb{R}^3$ is a vector space under these operations.

(a) $\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}+\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ and $r\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}rx\\ry\\rz\end{pmatrix}$

(b) $\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}+\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ and $r\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

✓ 1.28 For each, decide if it is a vector space; the intended operations are the natural ones.

(a) The *diagonal* 2×2 matrices

$$\{\begin{pmatrix}a&0\\0&b\end{pmatrix} \mid a,b\in\mathbb{R}\}$$

(b) This set of 2×2 matrices

$$\{\begin{pmatrix}x&x+y\\x+y&y\end{pmatrix} \mid x,y\in\mathbb{R}\}$$

(c) This set

$$\{\begin{pmatrix}x\\y\\z\\w\end{pmatrix}\in\mathbb{R}^4 \mid x+y+w=1\}$$

(d) The set of functions $\{f\colon\mathbb{R}\to\mathbb{R} \mid df/dx+2f=0\}$

(e) The set of functions $\{f\colon\mathbb{R}\to\mathbb{R} \mid df/dx+2f=1\}$

✓ 1.29 Prove or disprove that this is a vector space: the real-valued functions f of one real variable such that $f(7)=0$.

✓ 1.30 Show that the set $\mathbb{R}^+$ of positive reals is a vector space when we interpret '$x+y$' to mean the product of x and y (so that $2+3$ is 6), and we interpret '$r\cdot x$' as the r-th power of x.

1.31 Is $\{(x,y) \mid x,y\in\mathbb{R}\}$ a vector space under these operations?

(a) $(x_1,y_1)+(x_2,y_2)=(x_1+x_2,y_1+y_2)$ and $r\cdot(x,y)=(rx,y)$

(b) $(x_1,y_1)+(x_2,y_2)=(x_1+x_2,y_1+y_2)$ and $r\cdot(x,y)=(rx,0)$

1.32 Prove or disprove that this is a vector space: the set of polynomials of degree greater than or equal to two, along with the zero polynomial.

1.33 At this point "the same" is only an intuition, but nonetheless for each vector space identify the k for which the space is "the same" as $\mathbb{R}^k$.

(a) The 2×3 matrices under the usual operations

(b) The $n\times m$ matrices (under their usual operations)

(c) This set of 2×2 matrices

$$\{\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a, b, c \in \mathbb{R}\}$$

(d) This set of 2×2 matrices

$$\{\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a + b + c = 0\}$$

✓ 1.34 Using $\vec{+}$ to represent vector addition and $\vec{\cdot}$ for scalar multiplication, restate the definition of vector space.

✓ 1.35 Prove these.

(a) Any vector is the additive inverse of the additive inverse of itself.

(b) Vector addition left-cancels: if $\vec{v}, \vec{s}, \vec{t} \in V$ then $\vec{v} + \vec{s} = \vec{v} + \vec{t}$ implies that $\vec{s} = \vec{t}$.

1.36 The definition of vector spaces does not explicitly say that $\vec{0} + \vec{v} = \vec{v}$ (it instead says that $\vec{v} + \vec{0} = \vec{v}$). Show that it must nonetheless hold in any vector space.

✓ 1.37 Prove or disprove that this is a vector space: the set of all matrices, under the usual operations.

1.38 In a vector space every element has an additive inverse. Can some elements have two or more?

1.39 (a) Prove that every point, line, or plane thru the origin in $\mathbb{R}^3$ is a vector space under the inherited operations.

(b) What if it doesn't contain the origin?

✓ 1.40 Using the idea of a vector space we can easily reprove that the solution set of a homogeneous linear system has either one element or infinitely many elements. Assume that $\vec{v} \in V$ is not $\vec{0}$.

(a) Prove that $r \cdot \vec{v} = \vec{0}$ if and only if $r = 0$.

(b) Prove that $r_1 \cdot \vec{v} = r_2 \cdot \vec{v}$ if and only if $r_1 = r_2$.

(c) Prove that any nontrivial vector space is infinite.

(d) Use the fact that a nonempty solution set of a homogeneous linear system is a vector space to draw the conclusion.

1.41 Is this a vector space under the natural operations: the real-valued functions of one real variable that are differentiable?

1.42 A *vector space over the complex numbers* $\mathbb{C}$ has the same definition as a vector space over the reals except that scalars are drawn from $\mathbb{C}$ instead of from $\mathbb{R}$. Show that each of these is a vector space over the complex numbers. (Recall how complex numbers add and multiply: $(a_0 + a_1 i) + (b_0 + b_1 i) = (a_0 + b_0) + (a_1 + b_1)i$ and $(a_0 + a_1 i)(b_0 + b_1 i) = (a_0 b_0 - a_1 b_1) + (a_0 b_1 + a_1 b_0)i$.)

(a) The set of degree two polynomials with complex coefficients

(b) This set

$$\{\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \mid a, b \in \mathbb{C} \text{ and } a + b = 0 + 0i\}$$

1.43 Name a property shared by all of the $\mathbb{R}^n$'s but not listed as a requirement for a vector space.

✓ 1.44 (a) Prove that for any four vectors $\vec{v}_1, \ldots, \vec{v}_4 \in V$ we can associate their sum in any way without changing the result.

$$\begin{aligned} ((\vec{v}_1 + \vec{v}_2) + \vec{v}_3) + \vec{v}_4 &= (\vec{v}_1 + (\vec{v}_2 + \vec{v}_3)) + \vec{v}_4 = (\vec{v}_1 + \vec{v}_2) + (\vec{v}_3 + \vec{v}_4) \\ &= \vec{v}_1 + ((\vec{v}_2 + \vec{v}_3) + \vec{v}_4) = \vec{v}_1 + (\vec{v}_2 + (\vec{v}_3 + \vec{v}_4)) \end{aligned}$$

This allows us to write '$\vec{v}_1 + \vec{v}_2 + \vec{v}_3 + \vec{v}_4$' without ambiguity.

(b) Prove that any two ways of associating a sum of any number of vectors give the same sum. (*Hint.* Use induction on the number of vectors.)

1.45 Example 1.5 gives a subset of $\mathbb{R}^2$ that is not a vector space, under the obvious operations, because while it is closed under addition, it is not closed under scalar multiplication. Consider the set of vectors in the plane whose components have the same sign or are 0. Show that this set is closed under scalar multiplication but not addition.

1.46 For any vector space, a subset that is itself a vector space under the inherited operations (e.g., a plane through the origin inside of $\mathbb{R}^3$) is a *subspace*.

(a) Show that $\{a_0 + a_1x + a_2x^2 \mid a_0 + a_1 + a_2 = 0\}$ is a subspace of the vector space of degree two polynomials.

(b) Show that this is a subspace of the 2×2 matrices.

$$\{\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \mid a + b = 0\}$$

(c) Show that a nonempty subset S of a real vector space is a subspace if and only if it is closed under linear combinations of pairs of vectors: whenever $c_1, c_2 \in \mathbb{R}$ and $\vec{s}_1, \vec{s}_2 \in S$ then the combination $c_1\vec{v}_1 + c_2\vec{v}_2$ is in S.

I.2 Subspaces and Spanning Sets

One of the examples that led us to define vector spaces was the solution set of a homogeneous system. For instance, we saw in Example 1.4 such a space that is a planar subset of $\mathbb{R}^3$. There, the vector space $\mathbb{R}^3$ contains inside it another vector space, the plane.

2.1 Definition For any vector space, a *subspace* is a subset that is itself a vector space, under the inherited operations.

2.2 Example Example 1.4's plane

$$P = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x + y + z = 0 \}$$

is a subspace of $\mathbb{R}^3$. As required by the definition the plane's operations are inherited from the larger space, that is, vectors add in P as they add in $\mathbb{R}^3$

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$$

and scalar multiplication is also the same as in $\mathbb{R}^3$. To show that P is a subspace we need only note that it is a subset and then verify that it is a space. We have already checked in Example 1.4 that P satisfies the conditions in the definition of a vector space. For instance, for closure under addition we noted that if the summands satisfy that $x_1 + y_1 + z_1 = 0$ and $x_2 + y_2 + z_2 = 0$ then the sum satisfies that $(x_1+x_2)+(y_1+y_2)+(z_1+z_2) = (x_1+y_1+z_1)+(x_2+y_2+z_2) = 0$.

2.3 Example The x-axis in $\mathbb{R}^2$ is a subspace, where the addition and scalar multiplication operations are the inherited ones.

$$\begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ 0 \end{pmatrix} \qquad r \cdot \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} rx \\ 0 \end{pmatrix}$$

As in the prior example, to verify directly from the definition that this is a subspace we simply that it is a subset and then check that it satisfies the conditions in definition of a vector space. For instance the two closure conditions are satisfied: adding two vectors with a second component of zero results in a vector with a second component of zero and multiplying a scalar times a vector with a second component of zero results in a vector with a second component of zero.

2.4 Example Another subspace of $\mathbb{R}^2$ is its trivial subspace.

$$\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \}$$

Any vector space has a trivial subspace $\{\vec{0}\,\}$. At the opposite extreme, any vector space has itself for a subspace. These two are the *improper* subspaces. Other subspaces are *proper*.

2.5 Example Vector spaces that are not $\mathbb{R}^n$'s also have subspaces. The space of cubic polynomials $\{a + bx + cx^2 + dx^3 \mid a, b, c, d \in \mathbb{R}\}$ has a subspace comprised of all linear polynomials $\{m + nx \mid m, n \in \mathbb{R}\}$.

2.6 Example Another example of a subspace that is not a subset of an $\mathbb{R}^n$ followed the definition of a vector space. The space in Example 1.12 of all real-valued functions of one real variable $\{f \mid f\colon \mathbb{R} \to \mathbb{R}\}$ has the subspace in Example 1.14 of functions satisfying the restriction $(d^2 f/dx^2) + f = 0$.

2.7 Example The definition requires that the addition and scalar multiplication operations must be the ones inherited from the larger space. The set $S = \{1\}$ is a subset of $\mathbb{R}^1$. And, under the operations $1 + 1 = 1$ and $r \cdot 1 = 1$ the set S is a vector space, specifically, a trivial space. However, S is not a subspace of $\mathbb{R}^1$ because those aren't the inherited operations, since of course $\mathbb{R}^1$ has $1 + 1 = 2$.

2.8 Example Being vector spaces themselves, subspaces must satisfy the closure conditions. The set $\mathbb{R}^+$ is not a subspace of the vector space $\mathbb{R}^1$ because with the inherited operations it is not closed under scalar multiplication: if $\vec{v} = 1$ then $-1 \cdot \vec{v} \not\in \mathbb{R}^+$.

The next result says that Example 2.8 is prototypical. The only way that a subset can fail to be a subspace, if it is nonempty and uses the inherited operations, is if it isn't closed.

2.9 Lemma For a nonempty subset S of a vector space, under the inherited operations the following are equivalent statements.*

(1) S is a subspace of that vector space

(2) S is closed under linear combinations of pairs of vectors: for any vectors $\vec{s}_1, \vec{s}_2 \in S$ and scalars r_1, r_2 the vector $r_1\vec{s}_1 + r_2\vec{s}_2$ is in S

(3) S is closed under linear combinations of any number of vectors: for any vectors $\vec{s}_1, \ldots, \vec{s}_n \in S$ and scalars $r_1, \ldots, r_n$ the vector $r_1\vec{s}_1 + \cdots + r_n\vec{s}_n$ is an element of S.

Briefly, a subset is a subspace if and only if it is closed under linear combinations.

Proof 'The following are equivalent' means that each pair of statements are equivalent.

$$(1) \iff (2) \qquad (2) \iff (3) \qquad (3) \iff (1)$$

We will prove the equivalence by establishing that $(1) \implies (3) \implies (2) \implies (1)$. This strategy is suggested by the observation that the implications $(1) \implies (3)$ and $(3) \implies (2)$ are easy and so we need only argue that $(2) \implies (1)$.

Assume that S is a nonempty subset of a vector space V that is S closed under combinations of pairs of vectors. We will show that S is a vector space by checking the conditions.

The vector space definition has five conditions on addition. First, for closure under addition, if $\vec{s}_1, \vec{s}_2 \in S$ then $\vec{s}_1 + \vec{s}_2 \in S$, as it is a combination of a pair

*More information on equivalence of statements is in the appendix.

of vectors and we are assuming that S is closed under those. Second, for any $\vec{s}_1, \vec{s}_2 \in S$, because addition is inherited from V, the sum $\vec{s}_1 + \vec{s}_2$ in S equals the sum $\vec{s}_1 + \vec{s}_2$ in V, and that equals the sum $\vec{s}_2 + \vec{s}_1$ in V (because V is a vector space, its addition is commutative), and that in turn equals the sum $\vec{s}_2 + \vec{s}_1$ in S. The argument for the third condition is similar to that for the second. For the fourth, consider the zero vector of V and note that closure of S under linear combinations of pairs of vectors gives that $0 \cdot \vec{s} + 0 \cdot \vec{s} = \vec{0}$ is an element of S (where $\vec{s}$ is any member of the nonempty set S); checking that $\vec{0}$ acts under the inherited operations as the additive identity of S is easy. The fifth condition is satisfied because for any $\vec{s} \in S$, closure under linear combinations of pairs of vectors shows that $0 \cdot \vec{0} + (-1) \cdot \vec{s}$ is an element of S, and it is obviously the additive inverse of $\vec{s}$ under the inherited operations.

The verifications for the scalar multiplication conditions are similar; see Exercise 33. QED

We will usually verify that a subset is a subspace by checking that it satisfies statement (2).

2.10 Remark At the start of this chapter we introduced vector spaces as collections in which linear combinations "make sense." Theorem 2.9's statements (1)-(3) say that we can always make sense of an expression like $r_1\vec{s}_1 + r_2\vec{s}_2$ in that the vector described is in the set S.

As a contrast, consider the set T of two-tall vectors whose entries add to a number greater than or equal to zero. Here we cannot just write any linear combination such as $2\vec{t}_1 - 3\vec{t}_2$ and be confident the result is an element of T.

Lemma 2.9 suggests that a good way to think of a vector space is as a collection of unrestricted linear combinations. The next two examples take some spaces and recasts their descriptions to be in that form.

2.11 Example We can show that this plane through the origin subset of $\mathbb{R}^3$

$$S = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x - 2y + z = 0 \}$$

is a subspace under the usual addition and scalar multiplication operations of column vectors by checking that it is nonempty and closed under linear combinations of two vectors. But there is another way. Think of $x - 2y + z = 0$ as a one-equation linear system and parametrize it by expressing the leading variable in terms of the free variables $x = 2y - z$.

$$S = \{ \begin{pmatrix} 2y - z \\ y \\ z \end{pmatrix} \mid y, z \in \mathbb{R} \} = \{ y \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \mid y, z \in \mathbb{R} \} \qquad (*)$$

Now, to show that this is a subspace consider $r_1\vec{s}_1 + r_2\vec{s}_2$. Each $\vec{s}_i$ is a linear combination of the two vectors in $(*)$ so this is a linear combination of linear combinations.

$$r_1 \cdot (y_1 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + z_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}) + r_2 \cdot (y_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + z_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix})$$

The Linear Combination Lemma, Lemma One.III.2.3, shows that the total is a linear combination of the two vectors and so Theorem 2.9's statement (2) is satisfied.

2.12 Example This is a subspace of the 2×2 matrices $\mathcal{M}_{2\times 2}$.

$$L = \{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a + b + c = 0 \}$$

To parametrize, express the condition as $a = -b - c$.

$$L = \{ \begin{pmatrix} -b-c & 0 \\ b & c \end{pmatrix} \mid b, c \in \mathbb{R} \} = \{ b \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} + c \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \mid b, c \in \mathbb{R} \}$$

As above, we've described the subspace as a collection of unrestricted linear combinations. To show it is a subspace, note that a linear combination of vectors from L is a linear combination of linear combinations and so statement (2) is true.

2.13 Definition The *span* (or *linear closure*) of a nonempty subset S of a vector space is the set of all linear combinations of vectors from S.

$$[S] = \{ c_1\vec{s}_1 + \cdots + c_n\vec{s}_n \mid c_1, \ldots, c_n \in \mathbb{R} \text{ and } \vec{s}_1, \ldots, \vec{s}_n \in S \}$$

The span of the empty subset of a vector space is its trivial subspace.

No notation for the span is completely standard. The square brackets used here are common but so are 'span(S)' and 'sp(S)'.

2.14 Remark In Chapter One, after we showed that we can write the solution set of a homogeneous linear system as $\{ c_1\vec{\beta}_1 + \cdots + c_k\vec{\beta}_k \mid c_1, \ldots, c_k \in \mathbb{R} \}$, we described that as the set 'generated' by the $\vec{\beta}$'s. We now call that the span of $\{ \vec{\beta}_1, \ldots, \vec{\beta}_k \}$.

Recall also from that proof that the span of the empty set is defined to be the set $\{\vec{0}\}$ because of the convention that a trivial linear combination, a combination of zero-many vectors, adds to $\vec{0}$. Besides, defining the empty set's span to be the trivial subspace is convenient because it keeps results like the next one from needing exceptions for the empty set.

2.15 Lemma In a vector space, the span of any subset is a subspace.

Proof If the subset S is empty then by definition its span is the trivial subspace. If S is not empty then by Lemma 2.9 we need only check that the span $[S]$ is closed under linear combinations of pairs of elements. For a pair of vectors from that span, $\vec{v} = c_1\vec{s}_1 + \cdots + c_n\vec{s}_n$ and $\vec{w} = c_{n+1}\vec{s}_{n+1} + \cdots + c_m\vec{s}_m$, a linear combination

$$\begin{aligned} &p \cdot (c_1\vec{s}_1 + \cdots + c_n\vec{s}_n) + r \cdot (c_{n+1}\vec{s}_{n+1} + \cdots + c_m\vec{s}_m) \\ &\qquad = pc_1\vec{s}_1 + \cdots + pc_n\vec{s}_n + rc_{n+1}\vec{s}_{n+1} + \cdots + rc_m\vec{s}_m \end{aligned}$$

is a linear combination of elements of S and so is an element of $[S]$ (possibly some of the $\vec{s}_i$'s from $\vec{v}$ equal some of the $\vec{s}_j$'s from $\vec{w}$ but that does not matter). QED

The converse of the lemma holds: any subspace is the span of some set, because a subspace is obviously the span of itself, the set of all of its members. Thus a subset of a vector space is a subspace if and only if it is a span. This fits the intuition that a good way to think of a vector space is as a collection in which linear combinations are sensible.

Taken together, Lemma 2.9 and Lemma 2.15 show that the span of a subset S of a vector space is the smallest subspace containing all of the members of S.

2.16 Example In any vector space V, for any vector $\vec{v} \in V$, the set $\{r \cdot \vec{v} \mid r \in \mathbb{R}\}$ is a subspace of V. For instance, for any vector $\vec{v} \in \mathbb{R}^3$ the line through the origin containing that vector $\{k\vec{v} \mid k \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$. This is true even if $\vec{v}$ is the zero vector, in which case it is the degenerate line, the trivial subspace.

2.17 Example The span of this set is all of $\mathbb{R}^2$.

$$\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\}$$

We know that the span is some subspace of $\mathbb{R}^2$. To check that it is all of $\mathbb{R}^2$ we must show that any member of $\mathbb{R}^2$ is a linear combination of these two vectors. So we ask: for which vectors with real components x and y are there scalars c_1 and c_2 such that this holds?

$$c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix} \qquad (*)$$

Gauss's Method

$$\begin{array}{rl} c_1 + c_2 = x \\ c_1 - c_2 = y \end{array} \xrightarrow{-\rho_1+\rho_2} \begin{array}{rl} c_1 + c_2 &= x \\ -2c_2 &= -x + y \end{array}$$

with back substitution gives $c_2 = (x-y)/2$ and $c_1 = (x+y)/2$. This shows that for any x, y there are appropriate coefficients c_1, c_2 making $(*)$ true — we can write any element of $\mathbb{R}^2$ as a linear combination of the two given ones. For instance, for $x = 1$ and $y = 2$ the coefficients $c_2 = -1/2$ and $c_1 = 3/2$ will do.

Since spans are subspaces, and we know that a good way to understand a subspace is to parametrize its description, we can try to understand a set's span in that way.

2.18 Example Consider, in the vector space of quadratic polynomials $\mathcal{P}_2$, the span of the set $S = \{3x - x^2, 2x\}$. By the definition of span, it is the set of unrestricted linear combinations of the two $\{c_1(3x - x^2) + c_2(2x) \mid c_1, c_2 \in \mathbb{R}\}$. Clearly polynomials in this span must have a constant term of zero. Is that necessary condition also sufficient?

We are asking: for which members $a_2x^2 + a_1x + a_0$ of $\mathcal{P}_2$ are there c_1 and c_2 such that $a_2x^2 + a_1x + a_0 = c_1(3x - x^2) + c_2(2x)$? Polynomials are equal if and only if their coefficients are equal so we are looking for conditions on a_2, a_1, and a_0 necessary for that triple to be a solution of this system.

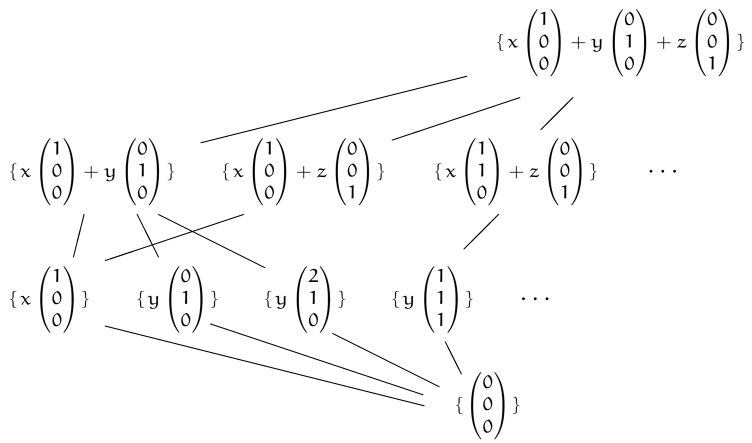
$$\begin{aligned} -c_1 &= a_2 \\ 3c_1 + 2c_2 &= a_1 \\ 0 &= a_0 \end{aligned}$$

Gauss's Method and back-substitution gives $c_1 = -a_2$, and $c_2 = (3/2)a_2 + (1/2)a_1$, and $0 = a_0$. Thus the only condition on elements $a_0 + a_1x + a_2x^2$ of the span is the condition that we knew: as long as there is no constant term $a_0 = 0$, we can give appropriate coefficients c_1 and c_2 to describe that polynomial as an element of the span. For instance, for the polynomial $0 - 4x + 3x^2$, the coefficients $c_1 = -3$ and $c_2 = 5/2$ will do. So the span of the given set is $[S] = \{a_1x + a_2x^2 \mid a_1, a_2 \in \mathbb{R}\}$.

Incidentally, this shows that the set $\{x, x^2\}$ spans the same subspace. A space can have more than one spanning set. Two other sets spanning this subspace are $\{x, x^2, -x + 2x^2\}$ and $\{x, x + x^2, x + 2x^2, \ldots\}$. (Usually we prefer to work with spanning sets that have only a small number of members.)

2.19 Example The picture below shows the subspaces of $\mathbb{R}^3$ that we now know of, the trivial subspace, the lines through the origin, the planes through the origin, and the whole space (of course, the picture shows only a few of the infinitely many subspaces). In the next section we will prove that $\mathbb{R}^3$ has no other type of subspaces, so in fact this picture shows them all.

That picture describes the subspaces as spans of sets with a minimal number of members. Note that the subspaces fall naturally into levels — planes on one level, lines on another, etc. — according to how many vectors are in the minimal-sized spanning set.



The line segments between levels connect subspaces with their superspaces.

So far in this chapter we have seen that to study the properties of linear combinations, the right setting is a collection that is closed under these combinations. In the first subsection we introduced such collections, vector spaces, and we saw a great variety of examples. In this subsection we saw still more spaces, ones that are subspaces of others. In all of the variety there is a commonality. Example 2.19 above brings it out: vector spaces and subspaces are best understood as a span, and especially as a span of a small number of vectors. The next section studies spanning sets that are minimal.

Exercises

✓ 2.20 Which of these subsets of the vector space of 2×2 matrices are subspaces under the inherited operations? For each one that is a subspace, parametrize its description. For each that is not, give a condition that fails.

(a) $\{\begin{pmatrix}a&0\\0&b\end{pmatrix} \mid a,b\in\mathbb{R}\}$

(b) $\{\begin{pmatrix}a&0\\0&b\end{pmatrix} \mid a+b=0\}$

(c) $\{\begin{pmatrix}a&0\\0&b\end{pmatrix} \mid a+b=5\}$

(d) $\{\begin{pmatrix}a&c\\0&b\end{pmatrix} \mid a+b=0, c\in\mathbb{R}\}$

✓ 2.21 Is this a subspace of $\mathcal{P}_2$: $\{a_0+a_1x+a_2x^2 \mid a_0+2a_1+a_2=4\}$? If it is then parametrize its description.

✓ 2.22 Decide if the vector lies in the span of the set, inside of the space.

(a) $\begin{pmatrix}2\\0\\1\end{pmatrix}$, $\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\}$, in $\mathbb{R}^3$

(b) $x-x^3$, $\{x^2, 2x+x^2, x+x^3\}$, in $\mathcal{P}_3$

(c) $\begin{pmatrix}0&1\\4&2\end{pmatrix}$, $\{\begin{pmatrix}1&0\\1&1\end{pmatrix},\begin{pmatrix}2&0\\2&3\end{pmatrix}\}$, in $\mathcal{M}_{2\times2}$

2.23 Which of these are members of the span $[\{\cos^2 x, \sin^2 x\}]$ in the vector space of real-valued functions of one real variable?

(a) $f(x) = 1$ **(b)** $f(x) = 3 + x^2$ **(c)** $f(x) = \sin x$ **(d)** $f(x) = \cos(2x)$

✓ 2.24 Which of these sets spans $\mathbb{R}^3$? That is, which of these sets has the property that any three-tall vector can be expressed as a suitable linear combination of the set's elements?

(a) $\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}0\\0\\3\end{pmatrix}\}$ **(b)** $\{\begin{pmatrix}2\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\}$ **(c)** $\{\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}3\\0\\0\end{pmatrix}\}$

(d) $\{\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}3\\1\\0\end{pmatrix},\begin{pmatrix}-1\\0\\0\end{pmatrix},\begin{pmatrix}2\\1\\5\end{pmatrix}\}$ **(e)** $\{\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}3\\0\\1\end{pmatrix},\begin{pmatrix}5\\1\\2\end{pmatrix},\begin{pmatrix}6\\0\\2\end{pmatrix}\}$

✓ 2.25 Parametrize each subspace's description. Then express each subspace as a span.

(a) The subset $\{(a\ \ b\ \ c) \mid a - c = 0\}$ of the three-wide row vectors

(b) This subset of $\mathcal{M}_{2\times2}$

$$\{\begin{pmatrix}a&b\\c&d\end{pmatrix} \mid a + d = 0\}$$

(c) This subset of $\mathcal{M}_{2\times2}$

$$\{\begin{pmatrix}a&b\\c&d\end{pmatrix} \mid 2a - c - d = 0 \text{ and } a + 3b = 0\}$$

(d) The subset $\{a + bx + cx^3 \mid a - 2b + c = 0\}$ of $\mathcal{P}_3$

(e) The subset of $\mathcal{P}_2$ of quadratic polynomials p such that $p(7) = 0$

✓ 2.26 Find a set to span the given subspace of the given space. (*Hint.* Parametrize each.)

(a) the xz-plane in $\mathbb{R}^3$

(b) $\{\begin{pmatrix}x\\y\\z\end{pmatrix} \mid 3x + 2y + z = 0\}$ in $\mathbb{R}^3$

(c) $\{\begin{pmatrix}x\\y\\z\\w\end{pmatrix} \mid 2x + y + w = 0 \text{ and } y + 2z = 0\}$ in $\mathbb{R}^4$

(d) $\{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_0 + a_1 = 0 \text{ and } a_2 - a_3 = 0\}$ in $\mathcal{P}_3$

(e) The set $\mathcal{P}_4$ in the space $\mathcal{P}_4$

(f) $\mathcal{M}_{2\times2}$ in $\mathcal{M}_{2\times2}$

2.27 Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$?

✓ 2.28 Decide if each is a subspace of the vector space of real-valued functions of one real variable.

(a) The *even* functions $\{f\colon \mathbb{R} \to \mathbb{R} \mid f(-x) = f(x) \text{ for all } x\}$. For example, two members of this set are $f_1(x) = x^2$ and $f_2(x) = \cos(x)$.

(b) The *odd* functions $\{f\colon \mathbb{R} \to \mathbb{R} \mid f(-x) = -f(x) \text{ for all } x\}$. Two members are $f_3(x) = x^3$ and $f_4(x) = \sin(x)$.

2.29 Example 2.16 says that for any vector $\vec{v}$ that is an element of a vector space V, the set $\{r \cdot \vec{v} \mid r \in \mathbb{R}\}$ is a subspace of V. (This is of course, simply the span of the singleton set $\{\vec{v}\}$.) Must any such subspace be a proper subspace, or can it be improper?

2.30 An example following the definition of a vector space shows that the solution set of a homogeneous linear system is a vector space. In the terminology of this subsection, it is a subspace of $\mathbb{R}^n$ where the system has n variables. What about a non-homogeneous linear system; do its solutions form a subspace (under the inherited operations)?

2.31 [Cleary] Give an example of each or explain why it would be impossible to do so.

(a) A nonempty subset of $\mathcal{M}_{2\times 2}$ that is not a subspace.

(b) A set of two vectors in $\mathbb{R}^2$ that does not span the space.

2.32 Example 2.19 shows that $\mathbb{R}^3$ has infinitely many subspaces. Does every non-trivial space have infinitely many subspaces?

2.33 Finish the proof of Lemma 2.9.

2.34 Show that each vector space has only one trivial subspace.

✓ 2.35 Show that for any subset S of a vector space, the span of the span equals the span $[[S]] = [S]$. (*Hint.* Members of $[S]$ are linear combinations of members of S. Members of $[[S]]$ are linear combinations of linear combinations of members of S.)

2.36 All of the subspaces that we've seen in some way use zero in their description. For example, the subspace in Example 2.3 consists of all the vectors from $\mathbb{R}^2$ with a second component of zero. In contrast, the collection of vectors from $\mathbb{R}^2$ with a second component of one does not form a subspace (it is not closed under scalar multiplication). Another example is Example 2.2, where the condition on the vectors is that the three components add to zero. If the condition there were that the three components add to one then it would not be a subspace (again, it would fail to be closed). However, a reliance on zero is not strictly necessary. Consider the set

$$\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x + y + z = 1\}$$

under these operations.

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 - 1 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} \qquad r\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rx - r + 1 \\ ry \\ rz \end{pmatrix}$$

(a) Show that it is not a subspace of $\mathbb{R}^3$. (*Hint.* See Example 2.7).

(b) Show that it is a vector space. Note that by the prior item, Lemma 2.9 can not apply.

(c) Show that any subspace of $\mathbb{R}^3$ must pass through the origin, and so any subspace of $\mathbb{R}^3$ must involve zero in its description. Does the converse hold? Does any subset of $\mathbb{R}^3$ that contains the origin become a subspace when given the inherited operations?

2.37 We can give a justification for the convention that the sum of zero-many vectors equals the zero vector. Consider this sum of three vectors $\vec{v}_1 + \vec{v}_2 + \vec{v}_3$.

(a) What is the difference between this sum of three vectors and the sum of the first two of these three?

(b) What is the difference between the prior sum and the sum of just the first one vector?

(c) What should be the difference between the prior sum of one vector and the sum of no vectors?

(d) So what should be the definition of the sum of no vectors?

2.38 Is a space determined by its subspaces? That is, if two vector spaces have the same subspaces, must the two be equal?

2.39 **(a)** Give a set that is closed under scalar multiplication but not addition.

(b) Give a set closed under addition but not scalar multiplication.

(c) Give a set closed under neither.

2.40 Show that the span of a set of vectors does not depend on the order in which the vectors are listed in that set.

2.41 Which trivial subspace is the span of the empty set? Is it

$$\{\begin{pmatrix}0\\0\\0\end{pmatrix}\} \subseteq \mathbb{R}^3, \quad \text{or} \quad \{0+0x\} \subseteq \mathcal{P}_1,$$

or some other subspace?

2.42 Show that if a vector is in the span of a set then adding that vector to the set won't make the span any bigger. Is that also 'only if'?

✓ 2.43 Subspaces are subsets and so we naturally consider how 'is a subspace of' interacts with the usual set operations.

(a) If A, B are subspaces of a vector space, must their intersection $A \cap B$ be a subspace? Always? Sometimes? Never?

(b) Must the union $A \cup B$ be a subspace?

(c) If A is a subspace, must its complement be a subspace?

(*Hint.* Try some test subspaces from Example 2.19.)

✓ 2.44 Does the span of a set depend on the enclosing space? That is, if W is a subspace of V and S is a subset of W (and so also a subset of V), might the span of S in W differ from the span of S in V?

2.45 Is the relation 'is a subspace of' transitive? That is, if V is a subspace of W and W is a subspace of X, must V be a subspace of X?

✓ 2.46 Because 'span of' is an operation on sets we naturally consider how it interacts with the usual set operations.

(a) If $S \subseteq T$ are subsets of a vector space, is $[S] \subseteq [T]$? Always? Sometimes? Never?

(b) If S, T are subsets of a vector space, is $[S \cup T] = [S] \cup [T]$?

(c) If S, T are subsets of a vector space, is $[S \cap T] = [S] \cap [T]$?

(d) Is the span of the complement equal to the complement of the span?

2.47 Reprove Lemma 2.15 without doing the empty set separately.

2.48 Find a structure that is closed under linear combinations, and yet is not a vector space.

II Linear Independence

The prior section shows how to understand a vector space as a span, as an unrestricted linear combination of some of its elements. For example, the space of linear polynomials $\{a + bx \mid a, b \in \mathbb{R}\}$ is spanned by the set $\{1, x\}$. The prior section also showed that a space can have many sets that span it. Two more sets that span the space of linear polynomials are $\{1, 2x\}$ and $\{1, x, 2x\}$.

At the end of that section we described some spanning sets as 'minimal' but we never precisely defined that word. We could mean that a spanning set is minimal if it contains the smallest number of members of any set with the same span, so that $\{1, x, 2x\}$ is not minimal because it has three members while we can give two-element sets spanning the same space. Or we could mean that a spanning set is minimal when it has no elements that we can remove without changing the span. Under this meaning $\{1, x, 2x\}$ is not minimal because removing the $2x$ to get $\{1, x\}$ leaves the span unchanged.

The first sense of minimality appears to be a global requirement, in that to check if a spanning set is minimal we seemingly must look at all the sets that span and find one with the least number of elements. The second sense of minimality is local since we need to look only at the set and consider the span with and without various elements. For instance, using the second sense we could compare the span of $\{1, x, 2x\}$ with the span of $\{1, x\}$ and note that $2x$ is a "repeat" in that its removal doesn't shrink the span.

In this section we will use the second sense of 'minimal spanning set' because of this technical convenience. However, the most important result of this book is that the two senses coincide. We will prove that in the next section.

II.1 Definition and Examples

We saw "repeats" in the first chapter. There, Gauss's Method turned them into $0 = 0$ equations.

1.1 Example Recall the Statics example from Chapter One's opening. We got two balances with the pair of unknown-mass objects, one at 40 cm and 15 cm and another at -50 cm and 25 cm, and we then computed the value of those masses. Had we instead gotten the second balance at 20 cm and 7.5 cm then Gauss's Method on the resulting two-equations, two-unknowns system would not have yielded a solution, it would have yielded a $0 = 0$ equation along with an equation containing a free variable. Intuitively, the problem is that $(20 \;\; 7.5)$ is half of $(40 \;\; 15)$, that is, $(20 \;\; 7.5)$ is in the span of the set $\{(40 \;\; 15)\}$ and so is repeated data. We would have been trying to solve a two-unknowns problem with essentially only one piece of information.

We take $\vec{v}$ to be a "repeat" of the vectors in a set S if $\vec{v} \in [S]$ so that it depends on, that is, is expressible in terms of, elements of the set $\vec{v} = c_1\vec{s}_1 + \cdots + c_n\vec{s}_n$.

1.2 Lemma Where V is a vector space, S is a subset of that space, and $\vec{v}$ is an element of that space, $[S \cup \{\vec{v}\}] = [S]$ if and only if $\vec{v} \in [S]$.

Proof Half of the if and only if is immediate: if $\vec{v} \notin [S]$ then the sets are not equal because $\vec{v} \in [S \cup \{\vec{v}\}]$.

For the other half assume that $\vec{v} \in [S]$ so that $\vec{v} = c_1\vec{s}_1 + \cdots + c_n\vec{s}_n$ for some scalars c_i and vectors $\vec{s}_i \in S$. We will use mutual containment to show that the sets $[S \cup \{\vec{v}\}]$ and $[S]$ are equal. The containment $[S \cup \{\vec{v}\}] \supseteq [S]$ is clear.

To show containment in the other direction let $\vec{w}$ be an element of $[S \cup \{\vec{v}\}]$. Then $\vec{w}$ is a linear combination of elements of $S \cup \{\vec{v}\}$, which we can write as $\vec{w} = c_{n+1}\vec{s}_{n+1} + \cdots + c_{n+k}\vec{s}_{n+k} + c_{n+k+1}\vec{v}$. (Possibly some of the $\vec{s}_i$'s from $\vec{w}$'s equation are the same as some of those from $\vec{v}$'s equation but that does not matter.) Expand $\vec{v}$.

$$\vec{w} = c_{n+1}\vec{s}_{n+1} + \cdots + c_{n+k}\vec{s}_{n+k} + c_{n+k+1} \cdot (c_1\vec{s}_1 + \cdots + c_n\vec{s}_n)$$

Recognize the right hand side as a linear combination of linear combinations of vectors from S. Thus $\vec{w} \in [S]$. QED

The discussion at the section's opening involved removing vectors instead of adding them.

1.3 Corollary For $\vec{v} \in S$, omitting that vector does not shrink the span $[S] = [S - \{\vec{v}\}]$ if and only if it is dependent on other vectors in the set $\vec{v} \in [S]$.

The corollary says that to know whether removing a vector will decrease the span, we need to know whether the vector is a linear combination of others in the set.

1.4 Definition A multiset subset of a vector space is *linearly independent* if none of its elements is a linear combination of the others.* Otherwise it is *linearly dependent*.

That definition's use of the word 'others' means that writing $\vec{v}$ as a linear combination with $\vec{v} = 1 \cdot \vec{v}$ does not count.

*More information on multisets is in the appendix. Most of the time we won't need the set-multiset distinction and we will follow the standard terminology of referring to a linearly independent or dependent 'set'. Remark 1.12 explains why the definition requires a multiset.

Observe that, although this way of writing one vector as a combination of the others

$$\vec{s}_0 = c_1\vec{s}_1 + c_2\vec{s}_2 + \cdots + c_n\vec{s}_n$$

visually sets off $\vec{s}_0$, algebraically there is nothing special about that vector in that equation. For any $\vec{s}_i$ with a coefficient c_i that is non-0 we can rewrite to isolate $\vec{s}_i$.

$$\vec{s}_i = (1/c_i)\vec{s}_0 + \cdots + (-c_{i-1}/c_i)\vec{s}_{i-1} + (-c_{i+1}/c_i)\vec{s}_{i+1} + \cdots + (-c_n/c_i)\vec{s}_n$$

When we don't want to single out any vector we will instead say that $\vec{s}_0, \vec{s}_1, \ldots, \vec{s}_n$ are in a *linear relationship* and put all of the vectors on the same side. The next result rephrases the linear independence definition in this style. It is how we usually compute whether a finite set is dependent or independent.

1.5 Lemma A subset S of a vector space is linearly independent if and only if among its elements the only linear relationship $c_1\vec{s}_1 + \cdots + c_n\vec{s}_n = \vec{0}$ (with $\vec{s}_i \neq \vec{s}_j$ for all $i \neq j$) is the trivial one $c_1 = 0, \ldots, c_n = 0$.

PROOF If S is linearly independent then no vector $\vec{s}_i$ is a linear combination of other vectors from S so there is no linear relationship where some of the $\vec{s}$'s have nonzero coefficients.

If S is not linearly independent then some $\vec{s}_i$ is a linear combination $\vec{s}_i = c_1\vec{s}_1 + \cdots + c_{i-1}\vec{s}_{i-1} + c_{i+1}\vec{s}_{i+1} + \cdots + c_n\vec{s}_n$ of other vectors from S. Subtracting $\vec{s}_i$ from both sides gives a relationship involving a nonzero coefficient, the -1 in front of $\vec{s}_i$. QED

1.6 Example In the vector space of two-wide row vectors, the two-element set $\{(40\ \ 15), (-50\ \ 25)\}$ is linearly independent. To check this, take

$$c_1 \cdot (40\ \ 15) + c_2 \cdot (-50\ \ 25) = (0\ \ 0)$$

and solve the resulting system.

$$\begin{array}{rl} 40c_1 - 50c_2 = 0 \\ 15c_1 + 25c_2 = 0 \end{array} \xrightarrow{-(15/40)\rho_1+\rho_2} \begin{array}{rl} 40c_1 - \quad 50c_2 = 0 \\ (175/4)c_2 = 0 \end{array}$$

Both c_1 and c_2 are zero. So the only linear relationship between the two given row vectors is the trivial relationship.

In the same vector space, the set $\{(40\ \ 15), (20\ \ 7.5)\}$ is linearly dependent since we can satisfy $c_1 \cdot (40\ \ 15) + c_2 \cdot (20\ \ 7.5) = (0\ \ 0)$ with $c_1 = 1$ and $c_2 = -2$.

1.7 Example The set $\{1 + x, 1 - x\}$ is linearly independent in $\mathcal{P}_2$, the space of quadratic polynomials with real coefficients, because

$$0 + 0x + 0x^2 = c_1(1 + x) + c_2(1 - x) = (c_1 + c_2) + (c_1 - c_2)x + 0x^2$$

gives

$$\begin{array}{rl} c_1 + c_2 = 0 \\ c_1 - c_2 = 0 \end{array} \xrightarrow{-\rho_1+\rho_2} \begin{array}{rl} c_1 + c_2 = 0 \\ 2c_2 = 0 \end{array}$$

since polynomials are equal only if their coefficients are equal. Thus, the only linear relationship between these two members of $\mathcal{P}_2$ is the trivial one.

1.8 Example The rows of this matrix

$$A = \begin{pmatrix} 2 & 3 & 1 & 0 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

form a linearly independent set. This is easy to check for this case but also recall that Lemma One.III.2.5 shows that the rows of any echelon form matrix form a linearly independent set.

1.9 Example In $\mathbb{R}^3$, where

$$\vec{v}_1 = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 2 \\ 9 \\ 2 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} 4 \\ 18 \\ 4 \end{pmatrix}$$

the set $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent because this is a relationship

$$0 \cdot \vec{v}_1 + 2 \cdot \vec{v}_2 - 1 \cdot \vec{v}_3 = \vec{0}$$

where not all of the scalars are zero (the fact that some of the scalars are zero doesn't matter).

That example illustrates why, although Definition 1.4 is a clearer statement of what independence means, Lemma 1.5 is better for computations. Working straight from the definition, someone trying to compute whether S is linearly independent would start by setting $\vec{v}_1 = c_2\vec{v}_2 + c_3\vec{v}_3$ and concluding that there are no such c_2 and c_3. But knowing that the first vector is not dependent on the other two is not enough. This person would have to go on to try $\vec{v}_2 = c_1\vec{v}_1 + c_3\vec{v}_3$, in order to find the dependence $c_1 = 0$, $c_3 = 1/2$. Lemma 1.5 gets the same conclusion with only one computation.

1.10 Example The empty subset of a vector space is linearly independent. There is no nontrivial linear relationship among its members as it has no members.

1.11 Example In any vector space, any subset containing the zero vector is linearly dependent. One example is, in the space $\mathcal{P}_2$ of quadratic polynomials, the subset $\{1 + x, x + x^2, 0\}$. It is linearly dependent because $0 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + 1 \cdot \vec{0} = \vec{0}$ is a nontrivial relationship, since not all of the coefficients are zero.

A subtler way to see that this subset is dependent is to remember that the zero vector is equal to the trivial sum, the sum of the empty set. So any set containing the zero vector has an element that is a combination of a subset of other vectors from the set, specifically, the zero vector is a combination of the empty subset.

1.12 Remark [Velleman] Definition 1.4 says that when we decide whether some S is linearly independent, we must consider it as a multiset. (Recall that in a set repeated elements collapse so the set $\{0,1,0\}$ equals the set $\{0,1\}$, whereas in a multiset they do not collapse so the multiset $\{0,1,0\}$ contains the element 0 twice.) Here is an example showing that we can need multiset rather than set. In the next chapter we will look at functions from one vector space to another. Let the function $f\colon \mathcal{P}_1 \to \mathbb{R}$ be $f(a+bx)=a$ so that for instance $f(1+2x)=1$. Consider the subset $B=\{1,1+x\}$ of the domain. The images of the elements are $f(1)=1$ and $f(1+x)=1$. Because in a set repeated elements collapse these images form a set with one element $\{1\}$, which is linearly independent. But in a multiset repeated elements do not collapse so these images form a linearly dependent multiset $\{1,1\}$. The second case is the correct one: B is linearly independent but its image under f is linearly dependent.

1.13 Corollary A set S is linearly independent if and only if for any $\vec{v}\in S$, its removal shrinks the span $[S-\{v\}]\subset[S]$.

PROOF This follows from Corollary 1.3. If S is linearly independent then none of its vectors is dependent on the other elements, so removal of any vector will shrink the span. If S is not linearly independent then it contains a vector that is dependent on other elements of the set, and removal of that vector will not shrink the span. QED

So a spanning set is minimal if and only if it is linearly independent.

The prior result addresses removing elements from a linearly independent set. The next one adds elements.

1.14 Lemma Suppose that S is linearly independent and that $\vec{v}\notin S$. Then the set $S\cup\{\vec{v}\}$ is linearly independent if and only if $\vec{v}\notin[S]$.

PROOF We will show that $S\cup\{\vec{v}\}$ is not linearly independent if and only if $\vec{v}\in[S]$.

Suppose first that $v\in[S]$. Express $\vec{v}$ as a combination $\vec{v}=c_1\vec{s}_1+\cdots+c_n\vec{s}_n$. Rewrite that $\vec{0}=c_1\vec{s}_1+\cdots+c_n\vec{s}_n-1\cdot\vec{v}$. Since $v\notin S$, it does not equal any of the $\vec{s}_i$ so this is a nontrivial linear dependence among the elements of $S\cup\{\vec{v}\}$. Thus that set is not linearly independent.

Now suppose that $S\cup\{\vec{v}\}$ is not linearly independent and consider a nontrivial dependence among its members $\vec{0}=c_1\vec{s}_1+\cdots+c_n\vec{s}_n+c_{n+1}\cdot\vec{v}$. If $c_{n+1}=0$

then that is a dependence among the elements of S, but we are assuming that S is independent, so $c_{n+1} \neq 0$. Rewrite the equation as $\vec{v} = (c_1/c_{n+1})\vec{s}_1 + \cdots + (c_n/c_{n+1})\vec{s}_n$ to get $\vec{v} \in [S]$ QED

1.15 Example This subset of $\mathbb{R}^3$ is linearly independent.

$$S = \{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \}$$

The span of S is the x-axis. Here are two supersets, one that is linearly dependent and the other independent.

$$\text{dependent: } \{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix} \} \qquad \text{independent: } \{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \}$$

We got the dependent superset by adding a vector from the x-axis and so the span did not grow. We got the independent superset by adding a vector that isn't in $[S]$, because it has a nonzero y component, causing the span to grow.

For the independent set

$$S = \{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \}$$

the span $[S]$ is the xy-plane. Here are two supersets.

$$\text{dependent: } \{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \} \qquad \text{independent: } \{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \}$$

As above, the additional member of the dependent superset comes from $[S]$, the xy-plane, while the added member of the independent superset comes from outside of that span.

Finally, consider this independent set

$$S = \{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \}$$

with $[S] = \mathbb{R}^3$. We can get a linearly dependent superset.

$$\text{dependent: } \{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \}$$

But there is no linearly independent superset os S. One way to see that is to note that for any vector that we would add to S, the equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

has a solution $c_1 = x$, $c_2 = y$, and $c_3 = z$. Another way to see it is that we cannot add any vectors from outside of the span $[S]$ because that span is $\mathbb{R}^3$.

1.16 Corollary In a vector space, any finite set has a linearly independent subset with the same span.

Proof If $S = \{\vec{s}_1, \ldots, \vec{s}_n\}$ is linearly independent then S itself satisfies the statement, so assume that it is linearly dependent.

By the definition of dependent, S contains a vector $\vec{v}_1$ that is a linear combination of the others. Define the set $S_1 = S - \{\vec{v}_1\}$. By Corollary 1.3 the span does not shrink $[S_1] = [S]$.

If S_1 is linearly independent then we are done. Otherwise iterate: take a vector $\vec{v}_2$ that is a linear combination of other members of S_1 and discard it to derive $S_2 = S_1 - \{\vec{v}_2\}$ such that $[S_2] = [S_1]$. Repeat this until a linearly independent set S_j appears; one must appear eventually because S is finite and the empty set is linearly independent. (Formally, this argument uses induction on the number of elements in S. Exercise 40 asks for the details.) QED

Thus if we have a set that is linearly dependent then we can, without changing the span, pare down by discarding what we have called "repeat" vectors.

1.17 Example This set spans $\mathbb{R}^3$ (the check is routine) but is not linearly independent.

$$S = \{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} \}$$

We will calculate which vectors to drop in order to get a subset that is independent but has the same span. This linear relationship

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + c_4 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + c_5 \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad (*)$$

gives a system

$$\begin{aligned} c_1 \quad\;\; + \;\; c_3 + \quad\;\; + 3c_5 &= 0 \\ 2c_2 + 2c_3 - c_4 + 3c_5 &= 0 \\ c_4 \qquad\;\; &= 0 \end{aligned}$$

whose solution set has this parametrization.

$$\{\begin{pmatrix}c_1\\c_2\\c_3\\c_4\\c_5\end{pmatrix}=c_3\begin{pmatrix}-1\\-1\\1\\0\\0\end{pmatrix}+c_5\begin{pmatrix}-3\\-3/2\\0\\0\\1\end{pmatrix}\mid c_3,c_5\in\mathbb{R}\}$$

Set $c_5=1$ and $c_3=0$ to get an instance of $(*)$.

$$-3\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}-\frac{3}{2}\cdot\begin{pmatrix}0\\2\\0\end{pmatrix}+0\cdot\begin{pmatrix}1\\2\\0\end{pmatrix}+0\cdot\begin{pmatrix}0\\-1\\1\end{pmatrix}+1\cdot\begin{pmatrix}3\\3\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

This shows that the vector from S that we've associated with c_5 is in the span of the set of c_1's vector and c_2's vector. We can discard S's fifth vector without shrinking the span.

Similarly, set $c_3=1$, and $c_5=0$ to get an instance of $(*)$ that shows we can discard S's third vector without shrinking the span. Thus this set has the same span as S.

$$\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}0\\-1\\1\end{pmatrix}\}$$

The check that it is linearly independent is routine.

1.18 Corollary A subset $S=\{\vec{s}_1,\dots,\vec{s}_n\}$ of a vector space is linearly dependent if and only if some $\vec{s_i}$ is a linear combination of the vectors $\vec{s}_1$, …, $\vec{s}_{i-1}$ listed before it.

Proof Consider $S_0=\{\}$, $S_1=\{\vec{s_1}\}$, $S_2=\{\vec{s}_1,\vec{s}_2\}$, etc. Some index $i\geqslant 1$ is the first one with $S_{i-1}\cup\{\vec{s}_i\}$ linearly dependent, and there $\vec{s}_i\in[S_{i-1}]$. QED

The proof of Corollary 1.16 describes producing a linearly independent set by shrinking, by taking subsets. And the proof of Corollary 1.18 describes finding a linearly dependent set by taking supersets. We finish this subsection by considering how linear independence and dependence interact with the subset relation between sets.

1.19 Lemma Any subset of a linearly independent set is also linearly independent. Any superset of a linearly dependent set is also linearly dependent.

Proof Both are clear. QED

Restated, subset preserves independence and superset preserves dependence.

Those are two of the four possible cases. The third case, whether subset preserves linear dependence, is covered by Example 1.17, which gives a linearly dependent set S with one subset that is linearly dependent and another that is independent. The fourth case, whether superset preserves linear independence, is covered by Example 1.15, which gives cases where a linearly independent set has both an independent and a dependent superset. This table summarizes.

	$\hat{S} \subset S$	$\hat{S} \supset S$
S *independent*	$\hat{S}$ must be independent	$\hat{S}$ may be either
S *dependent*	$\hat{S}$ may be either	$\hat{S}$ must be dependent

Example 1.15 has something else to say about the interaction between linear independence and superset. It names a linearly independent set that is maximal in that it has no supersets that are linearly independent. By Lemma 1.14 a linearly independent set is maximal if and only if it spans the entire space, because that is when all the vectors in the space are already in the span. This nicely complements Lemma 1.13, that a spanning set is minimal if and only if it is linearly independent.

Exercises

✓ **1.20** Decide whether each subset of $\mathbb{R}^3$ is linearly dependent or linearly independent.

(a) $\{\begin{pmatrix}1\\-3\\5\end{pmatrix}, \begin{pmatrix}2\\2\\4\end{pmatrix}, \begin{pmatrix}4\\-4\\14\end{pmatrix}\}$

(b) $\{\begin{pmatrix}1\\7\\7\end{pmatrix}, \begin{pmatrix}2\\7\\7\end{pmatrix}, \begin{pmatrix}3\\7\\7\end{pmatrix}\}$

(c) $\{\begin{pmatrix}0\\0\\-1\end{pmatrix}, \begin{pmatrix}1\\0\\4\end{pmatrix}\}$

(d) $\{\begin{pmatrix}9\\9\\0\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}, \begin{pmatrix}3\\5\\-4\end{pmatrix}, \begin{pmatrix}12\\12\\-1\end{pmatrix}\}$

✓ **1.21** Which of these subsets of $\mathcal{P}_3$ are linearly dependent and which are independent?

(a) $\{3 - x + 9x^2, 5 - 6x + 3x^2, 1 + 1x - 5x^2\}$

(b) $\{-x^2, 1 + 4x^2\}$

(c) $\{2 + x + 7x^2, 3 - x + 2x^2, 4 - 3x^2\}$

(d) $\{8 + 3x + 3x^2, x + 2x^2, 2 + 2x + 2x^2, 8 - 2x + 5x^2\}$

✓ **1.22** Prove that each set $\{f, g\}$ is linearly independent in the vector space of all functions from $\mathbb{R}^+$ to $\mathbb{R}$.

(a) $f(x) = x$ and $g(x) = 1/x$

(b) $f(x) = \cos(x)$ and $g(x) = \sin(x)$

(c) $f(x) = e^x$ and $g(x) = \ln(x)$

✓ 1.23 Which of these subsets of the space of real-valued functions of one real variable is linearly dependent and which is linearly independent? (We have abbreviated some constant functions; e.g., in the first item, the '2' stands for the constant function $f(x) = 2$.)

(a) $\{2, 4\sin^2(x), \cos^2(x)\}$ **(b)** $\{1, \sin(x), \sin(2x)\}$ **(c)** $\{x, \cos(x)\}$

(d) $\{(1+x)^2, x^2+2x, 3\}$ **(e)** $\{\cos(2x), \sin^2(x), \cos^2(x)\}$ **(f)** $\{0, x, x^2\}$

1.24 Does the equation $\sin^2(x)/\cos^2(x) = \tan^2(x)$ show that this set of functions $\{\sin^2(x), \cos^2(x), \tan^2(x)\}$ is a linearly dependent subset of the set of all real-valued functions with domain the interval $(-\pi/2..\pi/2)$ of real numbers between $-\pi/2$ and $\pi/2$)?

1.25 Is the xy-plane subset of the vector space $\mathbb{R}^3$ linearly independent?

✓ 1.26 Show that the nonzero rows of an echelon form matrix form a linearly independent set.

✓ 1.27 **(a)** Show that if the set $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent then so is the set $\{\vec{u}, \vec{u}+\vec{v}, \vec{u}+\vec{v}+\vec{w}\}$.

(b) What is the relationship between the linear independence or dependence of $\{\vec{u}, \vec{v}, \vec{w}\}$ and the independence or dependence of $\{\vec{u}-\vec{v}, \vec{v}-\vec{w}, \vec{w}-\vec{u}\}$?

1.28 Example 1.10 shows that the empty set is linearly independent.

(a) When is a one-element set linearly independent?

(b) How about a set with two elements?

1.29 In any vector space V, the empty set is linearly independent. What about all of V?

1.30 Show that if $\{\vec{x}, \vec{y}, \vec{z}\}$ is linearly independent then so are all of its proper subsets: $\{\vec{x}, \vec{y}\}$, $\{\vec{x}, \vec{z}\}$, $\{\vec{y}, \vec{z}\}$, $\{\vec{x}\}$,$\{\vec{y}\}$, $\{\vec{z}\}$, and $\{\,\}$. Is that 'only if' also?

1.31 **(a)** Show that this

$$S = \{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \}$$

is a linearly independent subset of $\mathbb{R}^3$.

(b) Show that

$$\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$$

is in the span of S by finding c_1 and c_2 giving a linear relationship.

$$c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$$

Show that the pair c_1, c_2 is unique.

(c) Assume that S is a subset of a vector space and that $\vec{v}$ is in $[S]$, so that $\vec{v}$ is a linear combination of vectors from S. Prove that if S is linearly independent then a linear combination of vectors from S adding to $\vec{v}$ is unique (that is, unique up to reordering and adding or taking away terms of the form $0 \cdot \vec{s}$). Thus S as a spanning set is minimal in this strong sense: each vector in $[S]$ is a combination of elements of S a minimum number of times — only once.

(d) Prove that it can happen when S is not linearly independent that distinct linear combinations sum to the same vector.

1.32 Prove that a polynomial gives rise to the zero function if and only if it is the zero polynomial. (*Comment.* This question is not a Linear Algebra matter but we often use the result. A polynomial gives rise to a function in the natural way: $x \mapsto c_n x^n + \cdots + c_1 x + c_0$.)

1.33 Return to Section 1.2 and redefine point, line, plane, and other linear surfaces to avoid degenerate cases.

1.34 **(a)** Show that any set of four vectors in $\mathbb{R}^2$ is linearly dependent.

(b) Is this true for any set of five? Any set of three?

(c) What is the most number of elements that a linearly independent subset of $\mathbb{R}^2$ can have?

✓ 1.35 Is there a set of four vectors in $\mathbb{R}^3$ such that any three form a linearly independent set?

1.36 Must every linearly dependent set have a subset that is dependent and a subset that is independent?

1.37 In $\mathbb{R}^4$ what is the biggest linearly independent set you can find? The smallest? The biggest linearly dependent set? The smallest? ('Biggest' and 'smallest' mean that there are no supersets or subsets with the same property.)

✓ 1.38 Linear independence and linear dependence are properties of sets. We can thus naturally ask how the properties of linear independence and dependence act with respect to the familiar elementary set relations and operations. In this body of this subsection we have covered the subset and superset relations. We can also consider the operations of intersection, complementation, and union.

(a) How does linear independence relate to intersection: can an intersection of linearly independent sets be independent? Must it be?

(b) How does linear independence relate to complementation?

(c) Show that the union of two linearly independent sets can be linearly independent.

(d) Show that the union of two linearly independent sets need not be linearly independent.

1.39 *Continued from prior exercise.* What is the interaction between the property of linear independence and the operation of union?

(a) We might conjecture that the union $S \cup T$ of linearly independent sets is linearly independent if and only if their spans have a trivial intersection $[S] \cap [T] = \{\vec{0}\}$. What is wrong with this argument for the 'if' direction of that conjecture? "If the union $S \cup T$ is linearly independent then the only solution to $c_1\vec{s}_1 + \cdots + c_n\vec{s}_n + d_1\vec{t}_1 + \cdots + d_m\vec{t}_m = \vec{0}$ is the trivial one $c_1 = 0, \ldots, d_m = 0$. So any member of the intersection of the spans must be the zero vector because in $c_1\vec{s}_1 + \cdots + c_n\vec{s}_n = d_1\vec{t}_1 + \cdots + d_m\vec{t}_m$ each scalar is zero."

(b) Give an example showing that the conjecture is false.

(c) Find linearly independent sets S and T so that the union of $S-(S\cap T)$ and $T-(S\cap T)$ is linearly independent, but the union $S\cup T$ is not linearly independent.

(d) Characterize when the union of two linearly independent sets is linearly independent, in terms of the intersection of spans.

✓ 1.40 For Corollary 1.16,

(a) fill in the induction for the proof;

(b) give an alternate proof that starts with the empty set and builds a sequence of linearly independent subsets of the given finite set until one appears with the same span as the given set.

1.41 With a some calculation we can get formulas to determine whether or not a set of vectors is linearly independent.

(a) Show that this subset of $\mathbb{R}^2$

$$\{\begin{pmatrix}a\\c\end{pmatrix},\begin{pmatrix}b\\d\end{pmatrix}\}$$

is linearly independent if and only if $ad-bc\neq 0$.

(b) Show that this subset of $\mathbb{R}^3$

$$\{\begin{pmatrix}a\\d\\g\end{pmatrix},\begin{pmatrix}b\\e\\h\end{pmatrix},\begin{pmatrix}c\\f\\i\end{pmatrix}\}$$

is linearly independent iff $aei+bfg+cdh-hfa-idb-gec\neq 0$.

(c) When is this subset of $\mathbb{R}^3$

$$\{\begin{pmatrix}a\\d\\g\end{pmatrix},\begin{pmatrix}b\\e\\h\end{pmatrix}\}$$

linearly independent?

(d) This is an opinion question: for a set of four vectors from $\mathbb{R}^4$, must there be a formula involving the sixteen entries that determines independence of the set? (You needn't produce such a formula, just decide if one exists.)

✓ 1.42 **(a)** Prove that a set of two perpendicular nonzero vectors from $\mathbb{R}^n$ is linearly independent when $n>1$.

(b) What if $n=1$? $n=0$?

(c) Generalize to more than two vectors.

1.43 Consider the set of functions from the interval $(-1\,\ldots\,1)\subseteq\mathbb{R}$ to $\mathbb{R}$.

(a) Show that this set is a vector space under the usual operations.

(b) Recall the formula for the sum of an infinite geometric series: $1+x+x^2+\cdots=1/(1-x)$ for all $x\in(-1..1)$. Why does this not express a dependence inside of the set $\{g(x)=1/(1-x), f_0(x)=1, f_1(x)=x, f_2(x)=x^2,\ldots\}$ (in the vector space that we are considering)? (*Hint.* Review the definition of linear combination.)

(c) Show that the set in the prior item is linearly independent.

This shows that some vector spaces exist with linearly independent subsets that are infinite.

1.44 Show that, where S is a subspace of V, if a subset T of S is linearly independent in S then T is also linearly independent in V. Is that 'only if'?

III Basis and Dimension

The prior section ends with the observation that a spanning set is minimal when it is linearly independent and a linearly independent set is maximal when it spans the space. So the notions of minimal spanning set and maximal independent set coincide. In this section we will name this idea and study its properties.

III.1 Basis

1.1 Definition A *basis* for a vector space is a sequence of vectors that is linearly independent and that spans the space.

Because a basis is a sequence, meaning that bases are different if they contain the same elements but in different orders, we denote it with angle brackets $\langle\vec{\beta}_1,\vec{\beta}_2,\ldots\rangle$.* (A sequence is linearly independent if the multiset consisting of the elements of the sequence in is independent. Similarly, a sequence spans the space if the set of elements of the sequence spans the space.)

1.2 Example This is a basis for $\mathbb{R}^2$.

$$\langle\begin{pmatrix}2\\4\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\rangle$$

It is linearly independent

$$c_1\begin{pmatrix}2\\4\end{pmatrix}+c_2\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\quad\Longrightarrow\quad\begin{aligned}2c_1+1c_2&=0\\4c_1+1c_2&=0\end{aligned}\quad\Longrightarrow\quad c_1=c_2=0$$

and it spans $\mathbb{R}^2$.

$$\begin{aligned}2c_1+1c_2&=x\\4c_1+1c_2&=y\end{aligned}\quad\Longrightarrow\quad c_2=2x-y\text{ and }c_1=(y-x)/2$$

1.3 Example This basis for $\mathbb{R}^2$ differs from the prior one

$$\langle\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}2\\4\end{pmatrix}\rangle$$

because it is in a different order. The verification that it is a basis is just as in the prior example.

* More information on sequences is in the appendix.

1.4 Example The space $\mathbb{R}^2$ has many bases. Another one is this.

$$\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle$$

The verification is easy.

1.5 Definition For any $\mathbb{R}^n$

$$\mathcal{E}_n = \langle \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \rangle$$

is the *standard* (or *natural*) basis. We denote these vectors $\vec{e}_1, \dots, \vec{e}_n$.

Calculus books denote $\mathbb{R}^2$'s standard basis vectors as $\vec{\imath}$ and $\vec{\jmath}$ instead of $\vec{e}_1$ and $\vec{e}_2$ and they denote to $\mathbb{R}^3$'s standard basis vectors as $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$ instead of $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$. Note that $\vec{e}_1$ means something different in a discussion of $\mathbb{R}^3$ than it means in a discussion of $\mathbb{R}^2$.

1.6 Example Consider the space $\{a \cdot \cos\theta + b \cdot \sin\theta \mid a, b \in \mathbb{R}\}$ of functions of the real variable θ. This is a natural basis $\langle \cos\theta, \sin\theta \rangle = \langle 1 \cdot \cos\theta + 0 \cdot \sin\theta, 0 \cdot \cos\theta + 1 \cdot \sin\theta \rangle$. A more generic basis for this space is $\langle \cos\theta - \sin\theta, 2\cos\theta + 3\sin\theta \rangle$. Verification that these two are bases is Exercise 25.

1.7 Example A natural basis for the vector space of cubic polynomials $\mathcal{P}_3$ is $\langle 1, x, x^2, x^3 \rangle$. Two other bases for this space are $\langle x^3, 3x^2, 6x, 6 \rangle$ and $\langle 1, 1+x, 1+x+x^2, 1+x+x^2+x^3 \rangle$. Checking that each is linearly independent and spans the space is easy.

1.8 Example The trivial space $\{\vec{0}\}$ has only one basis, the empty one $\langle \rangle$.

1.9 Example The space of finite-degree polynomials has a basis with infinitely many elements $\langle 1, x, x^2, \dots \rangle$.

1.10 Example We have seen bases before. In the first chapter we described the solution set of homogeneous systems such as this one

$$\begin{aligned} x + y \quad - w &= 0 \\ z + w &= 0 \end{aligned}$$

by parametrizing.

$$\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} y + \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} w \mid y, w \in \mathbb{R} \}$$

Thus the vector space of solutions is the span of a two-element set. This two-vector set is also linearly independent, which is easy to check. Therefore the solution set is a subspace of $\mathbb{R}^4$ with a basis comprised of these two vectors.

1.11 Example Parametrization finds bases for other vector spaces, not just for solution sets of homogeneous systems. To find a basis for this subspace of $\mathcal{M}_{2\times2}$

$$\{\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \mid a+b-2c=0\}$$

we rewrite the condition as $a=-b+2c$.

$$\{\begin{pmatrix} -b+2c & b \\ c & 0 \end{pmatrix} \mid b,c\in\mathbb{R}\}=\{b\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}+c\begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} \mid b,c\in\mathbb{R}\}$$

Thus, this is a natural candidate for a basis.

$$\langle\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}\rangle$$

The above work shows that it spans the space. Linear independence is also easy.

Consider again Example 1.2. To verify that the set spans the space we looked at linear combinations that total to a member of the space $c_1\vec{\beta}_1+c_2\vec{\beta}_2=\binom{x}{y}$. We only noted in that example that such a combination exists, that for each x,y there exists a c_1,c_2, but in fact the calculation also shows that the combination is unique: c_1 must be $(y-x)/2$ and c_2 must be $2x-y$.

1.12 Theorem In any vector space, a subset is a basis if and only if each vector in the space can be expressed as a linear combination of elements of the subset in one and only one way.

We consider linear combinations to be the same if they have the same summands but in a different order, or if they differ only in the addition or deletion of terms of the form '$0\cdot\vec{\beta}$'.

Proof A sequence is a basis if and only if its vectors form a set that spans and that is linearly independent. A subset is a spanning set if and only if each vector in the space is a linear combination of elements of that subset in at least one way. Thus we need only show that a spanning subset is linearly independent if and only if every vector in the space is a linear combination of elements from the subset in at most one way.

Consider two expressions of a vector as a linear combination of the members of the subset. Rearrange the two sums, and if necessary add some $0\cdot\vec{\beta}_i$

terms, so that the two sums combine the same $\vec{\beta}$'s in the same order: $\vec{v} = c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n$ and $\vec{v} = d_1\vec{\beta}_1 + d_2\vec{\beta}_2 + \cdots + d_n\vec{\beta}_n$. Now

$$c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n = d_1\vec{\beta}_1 + d_2\vec{\beta}_2 + \cdots + d_n\vec{\beta}_n$$

holds if and only if

$$(c_1 - d_1)\vec{\beta}_1 + \cdots + (c_n - d_n)\vec{\beta}_n = \vec{0}$$

holds. So, asserting that each coefficient in the lower equation is zero is the same thing as asserting that $c_i = d_i$ for each i, that is, that every vector is expressible as a linear combination of the $\vec{\beta}$'s in a unique way. QED

1.13 Definition In a vector space with basis B the *representation of* $\vec{v}$ *with respect to* B is the column vector of the coefficients used to express $\vec{v}$ as a linear combination of the basis vectors:

$$\mathrm{Rep}_B(\vec{v}) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

where $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ and $\vec{v} = c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n$. The c's are the *coordinates of* $\vec{v}$ *with respect to* B.

1.14 Example In $\mathcal{P}_3$, with respect to the basis $B = \langle 1, 2x, 2x^2, 2x^3 \rangle$, the representation of $x + x^2$ is

$$\mathrm{Rep}_B(x + x^2) = \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \\ 0 \end{pmatrix}_B$$

because $x + x^2 = 0 \cdot 1 + (1/2) \cdot 2x + (1/2) \cdot 2x^2 + 0 \cdot 2x^3$. With respect to a different basis $D = \langle 1 + x, 1 - x, x + x^2, x + x^3 \rangle$, the representation is different.

$$\mathrm{Rep}_D(x + x^2) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}_D$$

(When there is more than one basis around, to help keep straight which representation is with respect to which basis we often write it as a subscript on the column vector.)

1.15 Remark Definition 1.1 requires that a basis be a sequence because without that we couldn't write these coordinates in a fixed order.

1.16 Example In $\mathbb{R}^2$, where $\vec{v} = \binom{3}{2}$, to find the coordinates of that vector with respect to the basis

$$B = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \rangle$$

we solve

$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

and get that $c_1 = 3$ and $c_2 = -1/2$.

$$\operatorname{Rep}_B(\vec{v}) = \begin{pmatrix} 3 \\ -1/2 \end{pmatrix}$$

1.17 Remark This use of column notation and the term 'coordinate' has both a disadvantage and an advantage. The disadvantage is that representations look like vectors from $\mathbb{R}^n$, which can be confusing when the vector space is $\mathbb{R}^n$, as in the prior example. We must infer the intent from the context. For example, the phrase 'in $\mathbb{R}^2$, where $\vec{v} = \binom{3}{2}$' refers to the plane vector that, when in canonical position, ends at $(3, 2)$. And in the end of that example, although we've omitted a subscript B from the column, that the right side is a representation is clear from the context.

The advantage of the notation and the term is that they generalize the familiar case: in $\mathbb{R}^n$ and with respect to the standard basis $\mathcal{E}_n$, the vector starting at the origin and ending at $(v_1, \ldots, v_n)$ has this representation.

$$\operatorname{Rep}_{\mathcal{E}_n}(\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}) = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}_{\mathcal{E}_n}$$

Our main use of representations will come later but the definition appears here because the fact that every vector is a linear combination of basis vectors in a unique way is a crucial property of bases, and also to help make a point. For calculation of coordinates among other things, we shall restrict our attention to spaces with bases having only finitely many elements. That will start in the next subsection.

Exercises

1.18 Decide if each is a basis for $\mathcal{P}_2$.
(a) $\langle x^2 - x + 1, 2x + 1, 2x - 1 \rangle$ **(b)** $\langle x + x^2, x - x^2 \rangle$

✓ 1.19 Decide if each is a basis for $\mathbb{R}^3$.

(a) $\langle \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}3\\2\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \rangle$ **(b)** $\langle \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}3\\2\\1\end{pmatrix} \rangle$ **(c)** $\langle \begin{pmatrix}0\\2\\-1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\5\\0\end{pmatrix} \rangle$

(d) $\langle \begin{pmatrix}0\\2\\-1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\3\\0\end{pmatrix} \rangle$

✓ 1.20 Represent the vector with respect to the basis.

(a) $\begin{pmatrix}1\\2\end{pmatrix}$, $B = \langle \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix} \rangle \subseteq \mathbb{R}^2$

(b) $x^2 + x^3$, $D = \langle 1, 1+x, 1+x+x^2, 1+x+x^2+x^3 \rangle \subseteq \mathcal{P}_3$

(c) $\begin{pmatrix}0\\-1\\0\\1\end{pmatrix}$, $\mathcal{E}_4 \subseteq \mathbb{R}^4$

1.21 Find a basis for $\mathcal{P}_2$, the space of all quadratic polynomials. Must any such basis contain a polynomial of each degree: degree zero, degree one, and degree two?

1.22 Find a basis for the solution set of this system.

$$\begin{aligned} x_1 - 4x_2 + 3x_3 - x_4 &= 0 \\ 2x_1 - 8x_2 + 6x_3 - 2x_4 &= 0 \end{aligned}$$

✓ 1.23 Find a basis for $\mathcal{M}_{2\times 2}$, the space of 2×2 matrices.

✓ 1.24 Find a basis for each.

(a) The subspace $\{a_2x^2 + a_1x + a_0 \mid a_2 - 2a_1 = a_0\}$ of $\mathcal{P}_2$

(b) The space of three-wide row vectors whose first and second components add to zero

(c) This subspace of the 2×2 matrices

$$\{\begin{pmatrix}a & b\\0 & c\end{pmatrix} \mid c - 2b = 0\}$$

1.25 Check Example 1.6.

✓ 1.26 Find the span of each set and then find a basis for that span.

(a) $\{1+x, 1+2x\}$ in $\mathcal{P}_2$ **(b)** $\{2-2x, 3+4x^2\}$ in $\mathcal{P}_2$

✓ 1.27 Find a basis for each of these subspaces of the space $\mathcal{P}_3$ of cubic polynomials.

(a) The subspace of cubic polynomials $p(x)$ such that $p(7) = 0$

(b) The subspace of polynomials $p(x)$ such that $p(7) = 0$ and $p(5) = 0$

(c) The subspace of polynomials $p(x)$ such that $p(7) = 0$, $p(5) = 0$, and $p(3) = 0$

(d) The space of polynomials $p(x)$ such that $p(7) = 0$, $p(5) = 0$, $p(3) = 0$, and $p(1) = 0$

1.28 We've seen that the result of reordering a basis can be another basis. Must it be?

1.29 Can a basis contain a zero vector?

✓ 1.30 Let $\langle \vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3 \rangle$ be a basis for a vector space.

(a) Show that $\langle c_1\vec{\beta}_1, c_2\vec{\beta}_2, c_3\vec{\beta}_3 \rangle$ is a basis when $c_1, c_2, c_3 \neq 0$. What happens when at least one c_i is 0?

(b) Prove that $\langle \vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3 \rangle$ is a basis where $\vec{\alpha}_i = \vec{\beta}_1 + \vec{\beta}_i$.

1.31 Find one vector $\vec{v}$ that will make each into a basis for the space.

(a) $\langle\begin{pmatrix}1\\1\end{pmatrix},\vec{v}\rangle$ in $\mathbb{R}^2$ **(b)** $\langle\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\vec{v}\rangle$ in $\mathbb{R}^3$ **(c)** $\langle x, 1+x^2, \vec{v}\rangle$ in $\mathcal{P}_2$

✓ 1.32 Where $\langle\vec{\beta}_1,\dots,\vec{\beta}_n\rangle$ is a basis, show that in this equation

$$c_1\vec{\beta}_1+\dots+c_k\vec{\beta}_k=c_{k+1}\vec{\beta}_{k+1}+\dots+c_n\vec{\beta}_n$$

each of the c_i's is zero. Generalize.

1.33 A basis contains some of the vectors from a vector space; can it contain them all?

1.34 Theorem 1.12 shows that, with respect to a basis, every linear combination is unique. If a subset is not a basis, can linear combinations be not unique? If so, must they be?

✓ 1.35 A square matrix is *symmetric* if for all indices i and j, entry i,j equals entry j,i.

(a) Find a basis for the vector space of symmetric 2×2 matrices.

(b) Find a basis for the space of symmetric 3×3 matrices.

(c) Find a basis for the space of symmetric $n\times n$ matrices.

✓ 1.36 We can show that every basis for $\mathbb{R}^3$ contains the same number of vectors.

(a) Show that no linearly independent subset of $\mathbb{R}^3$ contains more than three vectors.

(b) Show that no spanning subset of $\mathbb{R}^3$ contains fewer than three vectors. *Hint:* recall how to calculate the span of a set and show that this method cannot yield all of $\mathbb{R}^3$ when we apply it to fewer than three vectors.

1.37 One of the exercises in the Subspaces subsection shows that the set

$$\{\begin{pmatrix}x\\y\\z\end{pmatrix} \mid x+y+z=1\}$$

is a vector space under these operations.

$$\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}+\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}=\begin{pmatrix}x_1+x_2-1\\y_1+y_2\\z_1+z_2\end{pmatrix} \qquad r\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}rx-r+1\\ry\\rz\end{pmatrix}$$

Find a basis.

III.2 Dimension

The previous subsection defines a basis of a vector space and shows that a space can have many different bases. So we cannot talk about "the" basis for a vector space. True, some vector spaces have bases that strike us as more natural than others, for instance, $\mathbb{R}^2$'s basis $\mathcal{E}_2$ or $\mathcal{P}_2$'s basis $\langle 1,x,x^2\rangle$. But for the vector space $\{a_2x^2+a_1x+a_0 \mid 2a_2-a_0=a_1\}$, no particular basis leaps out at us as the natural one. We cannot, in general, associate with a space any single basis that best describes it.

We can however find something about the bases that is uniquely associated with the space. This subsection shows that any two bases for a space have the same number of elements. So with each space we can associate a number, the number of vectors in any of its bases.

Before we start, we first limit our attention to spaces where at least one basis has only finitely many members.

2.1 Definition A vector space is *finite-dimensional* if it has a basis with only finitely many vectors.

One space that is not finite-dimensional is the set of polynomials with real coefficients Example 1.11; this space is not spanned by any finite subset since that would contain a polynomial of largest degree but this space has polynomials of all degrees. Such spaces are interesting and important but we will focus in a different direction. From now on we will study only finite-dimensional vector spaces. In the rest of this book we shall take 'vector space' to mean 'finite-dimensional vector space'.

2.2 Remark One reason for sticking to finite-dimensional spaces is so that the representation of a vector with respect to a basis is a finitely-tall vector and we can easily write it. Another reason is that the statement 'any infinite-dimensional vector space has a basis' is equivalent to a statement called the Axiom of Choice [Blass 1984] and so covering this would move us far past this book's scope. (A discussion of the Axiom of Choice is in the Frequently Asked Questions list for `sci.math`, and another accessible one is [Rucker].)

To prove the main theorem we shall use a technical result, the Exchange Lemma. We first illustrate it with an example.

2.3 Example Here is a basis for $\mathbb{R}^3$ and a vector given as a linear combination of members of that basis.

$$B = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \rangle \qquad \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = (-1) \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

Two of the basis vectors have non-zero coefficients. Pick one, for instance the first. Replace it with the vector that we've expressed as the combination

$$\hat{B} = \langle \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \rangle$$

and the result is another basis for $\mathbb{R}^3$.

2.4 Lemma (Exchange Lemma) Assume that $B = \langle \vec{\beta}_1, \dots, \vec{\beta}_n \rangle$ is a basis for a vector space, and that for the vector $\vec{v}$ the relationship $\vec{v} = c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n$ has $c_i \neq 0$. Then exchanging $\vec{\beta}_i$ for $\vec{v}$ yields another basis for the space.

Proof Call the outcome of the exchange $\hat{B} = \langle \vec{\beta}_1, \dots, \vec{\beta}_{i-1}, \vec{v}, \vec{\beta}_{i+1}, \dots, \vec{\beta}_n \rangle$.

We first show that $\hat{B}$ is linearly independent. Any relationship $d_1\vec{\beta}_1 + \cdots + d_i\vec{v} + \cdots + d_n\vec{\beta}_n = \vec{0}$ among the members of $\hat{B}$, after substitution for $\vec{v}$,

$$d_1\vec{\beta}_1 + \cdots + d_i \cdot (c_1\vec{\beta}_1 + \cdots + c_i\vec{\beta}_i + \cdots + c_n\vec{\beta}_n) + \cdots + d_n\vec{\beta}_n = \vec{0} \qquad (*)$$

gives a linear relationship among the members of B. The basis B is linearly independent so the coefficient d_ic_i of $\vec{\beta}_i$ is zero. Because we assumed that c_i is nonzero, $d_i = 0$. Using this in equation $(*)$ gives that all of the other d's are also zero. Therefore $\hat{B}$ is linearly independent.

We finish by showing that $\hat{B}$ has the same span as B. Half of this argument, that $[\hat{B}] \subseteq [B]$, is easy; we can write any member $d_1\vec{\beta}_1 + \cdots + d_i\vec{v} + \cdots + d_n\vec{\beta}_n$ of $[\hat{B}]$ as $d_1\vec{\beta}_1 + \cdots + d_i \cdot (c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n) + \cdots + d_n\vec{\beta}_n$, which is a linear combination of linear combinations of members of B, and hence is in $[B]$. For the $[B] \subseteq [\hat{B}]$ half of the argument, recall that if $\vec{v} = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$ with $c_i \neq 0$ then we can rearrange the equation to $\vec{\beta}_i = (-c_1/c_i)\vec{\beta}_1 + \cdots + (1/c_i)\vec{v} + \cdots + (-c_n/c_i)\vec{\beta}_n$. Now, consider any member $d_1\vec{\beta}_1 + \cdots + d_i\vec{\beta}_i + \cdots + d_n\vec{\beta}_n$ of $[B]$, substitute for $\vec{\beta}_i$ its expression as a linear combination of the members of $\hat{B}$, and recognize, as in the first half of this argument, that the result is a linear combination of linear combinations of members of $\hat{B}$, and hence is in $[\hat{B}]$. QED

2.5 Theorem In any finite-dimensional vector space, all bases have the same number of elements.

Proof Fix a vector space with at least one finite basis. Choose, from among all of this space's bases, one $B = \langle \vec{\beta}_1, \dots, \vec{\beta}_n \rangle$ of minimal size. We will show that any other basis $D = \langle \vec{\delta}_1, \vec{\delta}_2, \dots \rangle$ also has the same number of members, n. Because B has minimal size, D has no fewer than n vectors. We will argue that it cannot have more than n vectors.

The basis B spans the space and $\vec{\delta}_1$ is in the space, so $\vec{\delta}_1$ is a nontrivial linear combination of elements of B. By the Exchange Lemma, we can swap $\vec{\delta}_1$ for a vector from B, resulting in a basis B_1, where one element is $\vec{\delta}_1$ and all of the $n-1$ other elements are $\vec{\beta}$'s.

The prior paragraph forms the basis step for an induction argument. The inductive step starts with a basis B_k (for $1 \leqslant k < n$) containing k members of D and $n-k$ members of B. We know that D has at least n members so there is a $\vec{\delta}_{k+1}$. Represent it as a linear combination of elements of B_k. The key point: in that representation, at least one of the nonzero scalars must be associated with a $\vec{\beta}_i$ or else that representation would be a nontrivial linear relationship among elements of the linearly independent set D. Exchange $\vec{\delta}_{k+1}$ for $\vec{\beta}_i$ to get a new basis B_{k+1} with one $\vec{\delta}$ more and one $\vec{\beta}$ fewer than the previous basis B_k.

Repeat that until no $\vec{\beta}$'s remain, so that B_n contains $\vec{\delta}_1, \ldots, \vec{\delta}_n$. Now, D cannot have more than these n vectors because any $\vec{\delta}_{n+1}$ that remains would be in the span of B_n (since it is a basis) and hence would be a linear combination of the other $\vec{\delta}$'s, contradicting that D is linearly independent. QED

2.6 Definition The *dimension* of a vector space is the number of vectors in any of its bases.

2.7 Example Any basis for $\mathbb{R}^n$ has n vectors since the standard basis $\mathcal{E}_n$ has n vectors. Thus, this definition of 'dimension' generalizes the most familiar use of term, that $\mathbb{R}^n$ is n-dimensional.

2.8 Example The space $\mathcal{P}_n$ of polynomials of degree at most n has dimension $n+1$. We can show this by exhibiting any basis — $\langle 1, x, \ldots, x^n \rangle$ comes to mind — and counting its members.

2.9 Example The space of functions $\{a \cdot \cos\theta + b \cdot \sin\theta \mid a, b \in \mathbb{R}\}$ of the real variable θ has dimension 2 since this space has the basis $\langle \cos\theta, \sin\theta \rangle$.

2.10 Example A trivial space is zero-dimensional since its basis is empty.

Again, although we sometimes say 'finite-dimensional' for emphasis, from now on we take all vector spaces to be finite-dimensional. So in the next result the word 'space' means 'finite-dimensional vector space'.

2.11 Corollary No linearly independent set can have a size greater than the dimension of the enclosing space.

PROOF The proof of Theorem 2.5 never uses that D spans the space, only that it is linearly independent. QED

2.12 Example Recall the diagram from Example I.2.19 showing the subspaces of $\mathbb{R}^3$. Each subspace is described with a minimal spanning set, a basis. The whole space has a basis with three members, the plane subspaces have bases with two members, the line subspaces have bases with one member, and the trivial subspace has a basis with zero members. We could not in that section show that these are $\mathbb{R}^3$'s only subspaces. We can show it now. The prior corollary proves that the only subspaces of $\mathbb{R}^3$ are either three-, two-, one-, or zero-dimensional. There are no subspaces somehow, say, between lines and planes.

2.13 Corollary Any linearly independent set can be expanded to make a basis.

Proof If a linearly independent set is not already a basis then it must not span the space. Adding to the set a vector that is not in the span will preserve linear independence by Lemma II.1.14. Keep adding until the resulting set does span the space, which the prior corollary shows will happen after only a finite number of steps. QED

2.14 Corollary Any spanning set can be shrunk to a basis.

Proof Call the spanning set S. If S is empty then it is already a basis (the space must be a trivial space). If $S=\{\vec{0}\}$ then it can be shrunk to the empty basis, thereby making it linearly independent, without changing its span.

Otherwise, S contains a vector $\vec{s}_1$ with $\vec{s}_1 \neq \vec{0}$ and we can form a basis $B_1=\langle\vec{s}_1\rangle$. If $[B_1]=[S]$ then we are done. If not then there is a $\vec{s}_2 \in [S]$ such that $\vec{s}_2 \notin [B_1]$. Let $B_2=\langle\vec{s}_1,\vec{s}_2\rangle$; by Lemma II.1.14 this is linearly independent so if $[B_2]=[S]$ then we are done.

We can repeat this process until the spans are equal, which must happen in at most finitely many steps. QED

2.15 Corollary In an n-dimensional space, a set composed of n vectors is linearly independent if and only if it spans the space.

Proof First we will show that a subset with n vectors is linearly independent if and only if it is a basis. The 'if' is trivially true — bases are linearly independent. 'Only if' holds because a linearly independent set can be expanded to a basis, but a basis has n elements, so this expansion is actually the set that we began with.

To finish, we will show that any subset with n vectors spans the space if and only if it is a basis. Again, 'if' is trivial. 'Only if' holds because any spanning set can be shrunk to a basis, but a basis has n elements and so this shrunken set is just the one we started with. QED

The main result of this subsection, that all of the bases in a finite-dimensional vector space have the same number of elements, is the single most important result in this book. As Example 2.12 shows, it describes what vector spaces and subspaces there can be.

One immediate consequence brings us back to when we considered the two things that could be meant by the term 'minimal spanning set'. At that point we defined 'minimal' as linearly independent but we noted that another reasonable interpretation of the term is that a spanning set is 'minimal' when it has the fewest number of elements of any set with the same span. Now that we have shown that all bases have the same number of elements, we know that the two senses of 'minimal' are equivalent.

Exercises

Assume that all spaces are finite-dimensional unless otherwise stated.

✓ **2.16** Find a basis for, and the dimension of, $\mathcal{P}_2$.

2.17 Find a basis for, and the dimension of, the solution set of this system.

$$\begin{aligned} x_1 - 4x_2 + 3x_3 - x_4 &= 0 \\ 2x_1 - 8x_2 + 6x_3 - 2x_4 &= 0 \end{aligned}$$

✓ **2.18** Find a basis for, and the dimension of, $\mathcal{M}_{2\times2}$, the vector space of 2×2 matrices.

2.19 Find the dimension of the vector space of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

subject to each condition.

(a) $a, b, c, d \in \mathbb{R}$
(b) $a - b + 2c = 0$ and $d \in \mathbb{R}$
(c) $a + b + c = 0$, $a + b - c = 0$, and $d \in \mathbb{R}$

✓ **2.20** Find the dimension of each.
(a) The space of cubic polynomials $p(x)$ such that $p(7) = 0$
(b) The space of cubic polynomials $p(x)$ such that $p(7) = 0$ and $p(5) = 0$
(c) The space of cubic polynomials $p(x)$ such that $p(7) = 0$, $p(5) = 0$, and $p(3) = 0$
(d) The space of cubic polynomials $p(x)$ such that $p(7) = 0$, $p(5) = 0$, $p(3) = 0$, and $p(1) = 0$

2.21 What is the dimension of the span of the set $\{\cos^2\theta, \sin^2\theta, \cos 2\theta, \sin 2\theta\}$? This span is a subspace of the space of all real-valued functions of one real variable.

2.22 Find the dimension of $\mathbb{C}^{47}$, the vector space of 47-tuples of complex numbers.

2.23 What is the dimension of the vector space $\mathcal{M}_{3\times5}$ of 3×5 matrices?

✓ **2.24** Show that this is a basis for $\mathbb{R}^4$.

$$\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \rangle$$

(We can use the results of this subsection to simplify this job.)

2.25 Refer to Example 2.12.
(a) Sketch a similar subspace diagram for $\mathcal{P}_2$.
(b) Sketch one for $\mathcal{M}_{2\times2}$.

✓ **2.26** Where S is a set, the functions $f\colon S \to \mathbb{R}$ form a vector space under the natural operations: the sum $f + g$ is the function given by $f + g\,(s) = f(s) + g(s)$ and the scalar product is $r \cdot f\,(s) = r \cdot f(s)$. What is the dimension of the space resulting for each domain?

(a) $S = \{1\}$ (b) $S = \{1, 2\}$ (c) $S = \{1, \ldots, n\}$

2.27 (See Exercise 26.) Prove that this is an infinite-dimensional space: the set of all functions $f\colon \mathbb{R} \to \mathbb{R}$ under the natural operations.

2.28 (See Exercise 26.) What is the dimension of the vector space of functions $f\colon S \to \mathbb{R}$, under the natural operations, where the domain S is the empty set?

2.29 Show that any set of four vectors in $\mathbb{R}^2$ is linearly dependent.

2.30 Show that $\langle \vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3 \rangle \subset \mathbb{R}^3$ is a basis if and only if there is no plane through the origin containing all three vectors.

2.31 (a) Prove that any subspace of a finite dimensional space has a basis.
(b) Prove that any subspace of a finite dimensional space is finite dimensional.

2.32 Where is the finiteness of B used in Theorem 2.5?

✓ 2.33 Prove that if U and W are both three-dimensional subspaces of $\mathbb{R}^5$ then $U \cap W$ is non-trivial. Generalize.

2.34 A basis for a space consists of elements of that space. So we are naturally led to how the property 'is a basis' interacts with operations $\subseteq$ and $\cap$ and $\cup$. (Of course, a basis is actually a sequence in that it is ordered, but there is a natural extension of these operations.)
(a) Consider first how bases might be related by $\subseteq$. Assume that U, W are subspaces of some vector space and that $U \subseteq W$. Can there exist bases B_U for U and B_W for W such that $B_U \subseteq B_W$? Must such bases exist?
For any basis B_U for U, must there be a basis B_W for W such that $B_U \subseteq B_W$?
For any basis B_W for W, must there be a basis B_U for U such that $B_U \subseteq B_W$?
For any bases B_U, B_W for U and W, must B_U be a subset of B_W?
(b) Is the $\cap$ of bases a basis? For what space?
(c) Is the $\cup$ of bases a basis? For what space?
(d) What about the complement operation?

(*Hint.* Test any conjectures against some subspaces of $\mathbb{R}^3$.)

✓ 2.35 Consider how 'dimension' interacts with 'subset'. Assume U and W are both subspaces of some vector space, and that $U \subseteq W$.
(a) Prove that $\dim(U) \leqslant \dim(W)$.
(b) Prove that equality of dimension holds if and only if $U = W$.
(c) Show that the prior item does not hold if they are infinite-dimensional.

? 2.36 [Wohascum no. 47] For any vector $\vec{v}$ in $\mathbb{R}^n$ and any permutation σ of the numbers 1, 2, ..., n (that is, σ is a rearrangement of those numbers into a new order), define $\sigma(\vec{v})$ to be the vector whose components are $v_{\sigma(1)}$, $v_{\sigma(2)}$, ..., and $v_{\sigma(n)}$ (where $\sigma(1)$ is the first number in the rearrangement, etc.). Now fix $\vec{v}$ and let V be the span of $\{\sigma(\vec{v}) \mid \sigma \text{ permutes } 1, \ldots, n\}$. What are the possibilities for the dimension of V?

III.3 Vector Spaces and Linear Systems

We will now reconsider linear systems and Gauss's Method, aided by the tools and terms of this chapter. We will make three points.

For the first, recall the insight from the Chapter One that Gauss's Method works by taking linear combinations of rows — if two matrices are related by row operations $A \longrightarrow \cdots \longrightarrow B$ then each row of B is a linear combination of the rows of A. Therefore, the right setting in which to study row operations in general, and Gauss's Method in particular, is the following vector space.

3.1 Definition The *row space* of a matrix is the span of the set of its rows. The *row rank* is the dimension of this space, the number of linearly independent rows.

3.2 Example If

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$$

then $\mathrm{Rowspace}(A)$ is this subspace of the space of two-component row vectors.

$$\{c_1 \cdot (2\ \ 3) + c_2 \cdot (4\ \ 6) \mid c_1, c_2 \in \mathbb{R}\}$$

The second row vector is linearly dependent on the first and so we can simplify the above description to $\{c \cdot (2\ \ 3) \mid c \in \mathbb{R}\}$.

3.3 Lemma If two matrices A and B are related by a row operation

$$A \xrightarrow{\rho_i \leftrightarrow \rho_j} B \quad \text{or} \quad A \xrightarrow{k\rho_i} B \quad \text{or} \quad A \xrightarrow{k\rho_i + \rho_j} B$$

(for $i \neq j$ and $k \neq 0$) then their row spaces are equal. Hence, row-equivalent matrices have the same row space and therefore the same row rank.

PROOF Corollary One.III.2.4 shows that when $A \longrightarrow B$ then each row of B is a linear combination of the rows of A. That is, in the above terminology, each row of B is an element of the row space of A. Then $\mathrm{Rowspace}(B) \subseteq \mathrm{Rowspace}(A)$ follows because a member of the set $\mathrm{Rowspace}(B)$ is a linear combination of the rows of B, so it is a combination of combinations of the rows of A, and by the Linear Combination Lemma is also a member of $\mathrm{Rowspace}(A)$.

For the other set containment, recall Lemma One.III.1.5, that row operations are reversible so $A \longrightarrow B$ if and only if $B \longrightarrow A$. Then $\mathrm{Rowspace}(A) \subseteq \mathrm{Rowspace}(B)$ follows as in the previous paragraph. QED

Of course, Gauss's Method performs the row operations systematically, with the goal of echelon form.

3.4 Lemma The nonzero rows of an echelon form matrix make up a linearly independent set.

Proof Lemma One.III.2.5 says that no nonzero row of an echelon form matrix is a linear combination of the other rows. This result just restates that in this chapter's terminology. QED

Thus, in the language of this chapter, Gaussian reduction works by eliminating linear dependences among rows, leaving the span unchanged, until no nontrivial linear relationships remain among the nonzero rows. In short, Gauss's Method produces a basis for the row space.

3.5 Example From any matrix, we can produce a basis for the row space by performing Gauss's Method and taking the nonzero rows of the resulting echelon form matrix. For instance,

$$\begin{pmatrix} 1 & 3 & 1 \\ 1 & 4 & 1 \\ 2 & 0 & 5 \end{pmatrix} \xrightarrow[-2\rho_1+\rho_3]{-\rho_1+\rho_2} \xrightarrow{6\rho_2+\rho_3} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

produces the basis $\langle (1\ \ 3\ \ 1), (0\ \ 1\ \ 0), (0\ \ 0\ \ 3) \rangle$ for the row space. This is a basis for the row space of both the starting and ending matrices, since the two row spaces are equal.

Using this technique, we can also find bases for spans not directly involving row vectors.

3.6 Definition The *column space* of a matrix is the span of the set of its columns. The *column rank* is the dimension of the column space, the number of linearly independent columns.

Our interest in column spaces stems from our study of linear systems. An example is that this system

$$\begin{aligned} c_1 + 3c_2 + 7c_3 &= d_1 \\ 2c_1 + 3c_2 + 8c_3 &= d_2 \\ c_2 + 2c_3 &= d_3 \\ 4c_1 \qquad + 4c_3 &= d_4 \end{aligned}$$

has a solution if and only if the vector of d's is a linear combination of the other column vectors,

$$c_1 \begin{pmatrix} 1 \\ 2 \\ 0 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 3 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 7 \\ 8 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}$$

meaning that the vector of d's is in the column space of the matrix of coefficients.

3.7 Example Given this matrix,

$$\begin{pmatrix} 1 & 3 & 7 \\ 2 & 3 & 8 \\ 0 & 1 & 2 \\ 4 & 0 & 4 \end{pmatrix}$$

to get a basis for the column space, temporarily turn the columns into rows and reduce.

$$\begin{pmatrix} 1 & 2 & 0 & 4 \\ 3 & 3 & 1 & 0 \\ 7 & 8 & 2 & 4 \end{pmatrix} \xrightarrow[-7\rho_1+\rho_3]{-3\rho_1+\rho_2} \xrightarrow{-2\rho_2+\rho_3} \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 1 & -12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now turn the rows back to columns.

$$\langle \begin{pmatrix} 1 \\ 2 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 1 \\ -12 \end{pmatrix} \rangle$$

The result is a basis for the column space of the given matrix.

3.8 Definition The *transpose* of a matrix is the result of interchanging its rows and columns, so that column j of the matrix A is row j of A^{T} and vice versa.

So we can summarize the prior example as "transpose, reduce, and transpose back."

We can even, at the price of tolerating the as-yet-vague idea of vector spaces being "the same," use Gauss's Method to find bases for spans in other types of vector spaces.

3.9 Example To get a basis for the span of $\{x^2 + x^4, 2x^2 + 3x^4, -x^2 - 3x^4\}$ in the space $\mathcal{P}_4$, think of these three polynomials as "the same" as the row vectors $(0\ \ 0\ \ 1\ \ 0\ \ 1)$, $(0\ \ 0\ \ 2\ \ 0\ \ 3)$, and $(0\ \ 0\ \ -1\ \ 0\ \ -3)$, apply Gauss's Method

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & -1 & 0 & -3 \end{pmatrix} \xrightarrow[\rho_1+\rho_3]{-2\rho_1+\rho_2} \xrightarrow{2\rho_2+\rho_3} \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and translate back to get the basis $\langle x^2 + x^4, x^4 \rangle$. (As mentioned earlier, we will make the phrase "the same" precise at the start of the next chapter.)

Thus, the first point for this subsection is that the tools of this chapter give us a more conceptual understanding of Gaussian reduction.

For the second point observe that row operations on a matrix can change its column space.

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \xrightarrow{-2\rho_1+\rho_2} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

The column space of the left-hand matrix contains vectors with a second component that is nonzero but the column space of the right-hand matrix contains only vectors whose second component is zero, so the two spaces are different. This observation makes next result surprising.

3.10 Lemma Row operations do not change the column rank.

Proof Restated, if A reduces to B then the column rank of B equals the column rank of A.

This proof will be finished if we show that row operations do not affect linear relationships among columns, because the column rank is the size of the largest set of unrelated columns. That is, we will show that a relationship exists among columns (such as that the fifth column is twice the second plus the fourth) if and only if that relationship exists after the row operation. But this is exactly the first theorem of this book, Theorem One.I.1.5: in a relationship among columns,

$$c_1 \cdot \begin{pmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{m,1} \end{pmatrix} + \cdots + c_n \cdot \begin{pmatrix} a_{1,n} \\ a_{2,n} \\ \vdots \\ a_{m,n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

row operations leave unchanged the set of solutions $(c_1, \ldots, c_n)$. QED

Besides the prior result another way to make the point that Gauss's Method has something to say about the column space as well as about the row space is with Gauss-Jordan reduction. Recall that it ends with the reduced echelon form of a matrix, as here.

$$\begin{pmatrix} 1 & 3 & 1 & 6 \\ 2 & 6 & 3 & 16 \\ 1 & 3 & 1 & 6 \end{pmatrix} \longrightarrow \cdots \longrightarrow \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Consider the row space and the column space of this result.

The first point made earlier in this subsection says that to get a basis for the row space we can just collect the rows with leading entries. However, because this is in reduced echelon form, a basis for the column space is just as easy: collect the columns containing the leading entries, $\langle\vec{e}_1, \vec{e}_2\rangle$. Thus, for a reduced echelon form matrix we can find bases for the row and column spaces in essentially the same way, by taking the parts of the matrix, the rows or columns, containing the leading entries.

3.11 Theorem For any matrix, the row rank and column rank are equal.

Proof Bring the matrix to reduced echelon form. Then the row rank equals the number of leading entries since that equals the number of nonzero rows. Then also, the number of leading entries equals the column rank because the set of columns containing leading entries consists of some of the $\vec{e}_i$'s from a standard basis, and that set is linearly independent and spans the set of columns. Hence, in the reduced echelon form matrix, the row rank equals the column rank, because each equals the number of leading entries.

But Lemma 3.3 and Lemma 3.10 show that the row rank and column rank are not changed by using row operations to get to reduced echelon form. Thus the row rank and the column rank of the original matrix are also equal. QED

3.12 Definition The *rank* of a matrix is its row rank or column rank.

So the second point that we have made in this subsection is that the column space and row space of a matrix have the same dimension.

Our final point is that the concepts that we've seen arising naturally in the study of vector spaces are exactly the ones that we have studied with linear systems.

3.13 Theorem For linear systems with n unknowns and with matrix of coefficients A, the statements

(1) the rank of A is r

(2) the vector space of solutions of the associated homogeneous system has dimension $n - r$

are equivalent.

So if the system has at least one particular solution then for the set of solutions, the number of parameters equals $n - r$, the number of variables minus the rank of the matrix of coefficients.

Proof The rank of A is r if and only if Gaussian reduction on A ends with r nonzero rows. That's true if and only if echelon form matrices row equivalent to A have r-many leading variables. That in turn holds if and only if there are $n - r$ free variables. QED

3.14 Corollary Where the matrix A is $n \times n$, these statements

(1) the rank of A is n

(2) A is nonsingular

(3) the rows of A form a linearly independent set

(4) the columns of A form a linearly independent set

(5) any linear system whose matrix of coefficients is A has one and only one solution

are equivalent.

PROOF Clearly (1) $\iff$ (2) $\iff$ (3) $\iff$ (4). The last, (4) $\iff$ (5), holds because a set of n column vectors is linearly independent if and only if it is a basis for $\mathbb{R}^n$, but the system

$$c_1 \begin{pmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{m,1} \end{pmatrix} + \cdots + c_n \begin{pmatrix} a_{1,n} \\ a_{2,n} \\ \vdots \\ a_{m,n} \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{pmatrix}$$

has a unique solution for all choices of $d_1, \ldots, d_n \in \mathbb{R}$ if and only if the vectors of a's on the left form a basis. QED

3.15 Remark [Munkres] Sometimes the results of this subsection are mistakenly remembered to say that the general solution of an n unknowns system of m equations uses $n-m$ parameters. The number of equations is not the relevant figure, rather, what matters is the number of independent equations, the number of equations in a maximal independent set. Where there are r independent equations, the general solution involves $n-r$ parameters.

Exercises

3.16 Transpose each.

(a) $\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$ **(b)** $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ **(c)** $\begin{pmatrix} 1 & 4 & 3 \\ 6 & 7 & 8 \end{pmatrix}$ **(d)** $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

(e) $\begin{pmatrix} -1 & -2 \end{pmatrix}$

✓ 3.17 Decide if the vector is in the row space of the matrix.

(a) $\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \end{pmatrix}$ **(b)** $\begin{pmatrix} 0 & 1 & 3 \\ -1 & 0 & 1 \\ -1 & 2 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$

✓ 3.18 Decide if the vector is in the column space.

(a) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1 & 3 & 1 \\ 2 & 0 & 4 \\ 1 & -3 & -3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

✓ 3.19 Decide if the vector is in the column space of the matrix.

(a) $\begin{pmatrix}2&1\\2&5\end{pmatrix}, \begin{pmatrix}1\\-3\end{pmatrix}$ **(b)** $\begin{pmatrix}4&-8\\2&-4\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$ **(c)** $\begin{pmatrix}1&-1&1\\1&1&-1\\-1&-1&1\end{pmatrix}, \begin{pmatrix}2\\0\\0\end{pmatrix}$

✓ 3.20 Find a basis for the row space of this matrix.

$$\begin{pmatrix}2&0&3&4\\0&1&1&-1\\3&1&0&2\\1&0&-4&-1\end{pmatrix}$$

✓ 3.21 Find the rank of each matrix.

(a) $\begin{pmatrix}2&1&3\\1&-1&2\\1&0&3\end{pmatrix}$ **(b)** $\begin{pmatrix}1&-1&2\\3&-3&6\\-2&2&-4\end{pmatrix}$ **(c)** $\begin{pmatrix}1&3&2\\5&1&1\\6&4&3\end{pmatrix}$

(d) $\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$

✓ 3.22 Find a basis for the span of each set.

(a) $\{(1\ \ 3), (-1\ \ 3), (1\ \ 4), (2\ \ 1)\} \subseteq \mathcal{M}_{1\times 2}$

(b) $\{\begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}3\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\-3\\-3\end{pmatrix}\} \subseteq \mathbb{R}^3$

(c) $\{1+x, 1-x^2, 3+2x-x^2\} \subseteq \mathcal{P}_3$

(d) $\{\begin{pmatrix}1&0&1\\3&1&-1\end{pmatrix}, \begin{pmatrix}1&0&3\\2&1&4\end{pmatrix}, \begin{pmatrix}-1&0&-5\\-1&-1&-9\end{pmatrix}\} \subseteq \mathcal{M}_{2\times 3}$

3.23 Which matrices have rank zero? Rank one?

✓ 3.24 Given $a, b, c \in \mathbb{R}$, what choice of d will cause this matrix to have the rank of one?

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}$$

3.25 Find the column rank of this matrix.

$$\begin{pmatrix}1&3&-1&5&0&4\\2&0&1&0&4&1\end{pmatrix}$$

3.26 Show that a linear system with at least one solution has at most one solution if and only if the matrix of coefficients has rank equal to the number of its columns.

✓ 3.27 If a matrix is 5×9, which set must be dependent, its set of rows or its set of columns?

3.28 Give an example to show that, despite that they have the same dimension, the row space and column space of a matrix need not be equal. Are they ever equal?

3.29 Show that the set $\{(1,-1,2,-3), (1,1,2,0), (3,-1,6,-6)\}$ does not have the same span as $\{(1,0,1,0), (0,2,0,3)\}$. What, by the way, is the vector space?

✓ 3.30 Show that this set of column vectors

$$\{\begin{pmatrix}d_1\\d_2\\d_3\end{pmatrix} \mid \text{there are } x, y, \text{ and } z \text{ such that: } \begin{aligned}3x+2y+4z&=d_1\\ x-z&=d_2\\ 2x+2y+5z&=d_3\end{aligned}\}$$

is a subspace of $\mathbb{R}^3$. Find a basis.

3.31 Show that the transpose operation is linear:

$$(rA + sB)^{\mathsf{T}} = rA^{\mathsf{T}} + sB^{\mathsf{T}}$$

for $r, s \in \mathbb{R}$ and $A, B \in \mathcal{M}_{m \times n}$.

✓ 3.32 In this subsection we have shown that Gaussian reduction finds a basis for the row space.

(a) Show that this basis is not unique — different reductions may yield different bases.

(b) Produce matrices with equal row spaces but unequal numbers of rows.

(c) Prove that two matrices have equal row spaces if and only if after Gauss-Jordan reduction they have the same nonzero rows.

3.33 Why is there not a problem with Remark 3.15 in the case that r is bigger than n?

3.34 Show that the row rank of an $m \times n$ matrix is at most m. Is there a better bound?

✓ 3.35 Show that the rank of a matrix equals the rank of its transpose.

3.36 True or false: the column space of a matrix equals the row space of its transpose.

✓ 3.37 We have seen that a row operation may change the column space. Must it?

3.38 Prove that a linear system has a solution if and only if that system's matrix of coefficients has the same rank as its augmented matrix.

3.39 An $m \times n$ matrix has *full row rank* if its row rank is m, and it has *full column rank* if its column rank is n.

(a) Show that a matrix can have both full row rank and full column rank only if it is square.

(b) Prove that the linear system with matrix of coefficients A has a solution for any $d_1, \ldots, d_n$'s on the right side if and only if A has full row rank.

(c) Prove that a homogeneous system has a unique solution if and only if its matrix of coefficients A has full column rank.

(d) Prove that the statement "if a system with matrix of coefficients A has any solution then it has a unique solution" holds if and only if A has full column rank.

3.40 How would the conclusion of Lemma 3.3 change if Gauss's Method were changed to allow multiplying a row by zero?

✓ 3.41 What is the relationship between $\text{rank}(A)$ and $\text{rank}(-A)$? Between $\text{rank}(A)$ and $\text{rank}(kA)$? What, if any, is the relationship between $\text{rank}(A)$, $\text{rank}(B)$, and $\text{rank}(A + B)$?

III.4 Combining Subspaces

This subsection is optional. It is required only for the last sections of Chapter Three and Chapter Five and for occasional exercises. You can pass it over without loss of continuity.

One way to understand something is to see how to build it from component parts. For instance, we sometimes think of $\mathbb{R}^3$ put together from the x-axis, the y-axis, and z-axis. In this subsection we will describe how to decompose a vector space into a combination of some of its subspaces. In developing this idea of subspace combination, we will keep the $\mathbb{R}^3$ example in mind as a prototype.

Subspaces are subsets and sets combine via union. But taking the combination operation for subspaces to be the simple set union operation isn't what we want. For instance, the union of the x-axis, the y-axis, and z-axis is not all of $\mathbb{R}^3$. In fact this union is not a subspace because it is not closed under addition: this vector

$$\begin{pmatrix}1\\0\\0\end{pmatrix}+\begin{pmatrix}0\\1\\0\end{pmatrix}+\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

is in none of the three axes and hence is not in the union. Therefore to combine subspaces, in addition to the members of those subspaces, we must at least also include all of their linear combinations.

4.1 Definition Where $W_1,\ldots,W_k$ are subspaces of a vector space, their *sum* is the span of their union $W_1+W_2+\cdots+W_k=[W_1\cup W_2\cup\cdots W_k]$.

Writing '+' fits with the conventional practice of using this symbol for a natural accumulation operation.

4.2 Example Our $\mathbb{R}^3$ prototype works with this. Any vector $\vec{w}\in\mathbb{R}^3$ is a linear combination $c_1\vec{v}_1+c_2\vec{v}_2+c_3\vec{v}_3$ where $\vec{v}_1$ is a member of the x-axis, etc., in this way

$$\begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix}=1\cdot\begin{pmatrix}w_1\\0\\0\end{pmatrix}+1\cdot\begin{pmatrix}0\\w_2\\0\end{pmatrix}+1\cdot\begin{pmatrix}0\\0\\w_3\end{pmatrix}$$

and so x-axis $+$ y-axis $+$ z-axis $=\mathbb{R}^3$.

4.3 Example A sum of subspaces can be less than the entire space. Inside of $\mathcal{P}_4$, let L be the subspace of linear polynomials $\{a+bx \mid a,b\in\mathbb{R}\}$ and let C be the subspace of purely-cubic polynomials $\{cx^3 \mid c\in\mathbb{R}\}$. Then $L+C$ is not all of $\mathcal{P}_4$. Instead, $L+C=\{a+bx+cx^3 \mid a,b,c\in\mathbb{R}\}$.

4.4 Example A space can be described as a combination of subspaces in more than one way. Besides the decomposition $\mathbb{R}^3=x$-axis $+$ y-axis $+$ z-axis, we can also write $\mathbb{R}^3=xy$-plane $+$ yz-plane. To check this, note that any $\vec{w}\in\mathbb{R}^3$ can be written as a linear combination of a member of the xy-plane and a member of the yz-plane; here are two such combinations.

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = 1 \cdot \begin{pmatrix} w_1 \\ w_2 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 0 \\ w_3 \end{pmatrix} \qquad \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = 1 \cdot \begin{pmatrix} w_1 \\ w_2/2 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ w_2/2 \\ w_3 \end{pmatrix}$$

The above definition gives one way in which we can think of a space as a combination of some of its parts. However, the prior example shows that there is at least one interesting property of our benchmark model that is not captured by the definition of the sum of subspaces. In the familiar decomposition of $\mathbb{R}^3$, we often speak of a vector's 'x part' or 'y part' or 'z part'. That is, in our prototype each vector has a unique decomposition into pieces from the parts making up the whole space. But in the decomposition used in Example 4.4, we cannot refer to the "xy part" of a vector — these three sums

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

all describe the vector as comprised of something from the first plane plus something from the second plane, but the "xy part" is different in each.

That is, when we consider how $\mathbb{R}^3$ is put together from the three axes we might mean "in such a way that every vector has at least one decomposition," which gives the definition above. But if we take it to mean "in such a way that every vector has one and only one decomposition" then we need another condition on combinations. To see what this condition is, recall that vectors are uniquely represented in terms of a basis. We can use this to break a space into a sum of subspaces such that any vector in the space breaks uniquely into a sum of members of those subspaces.

4.5 Example Consider $\mathbb{R}^3$ with its standard basis $\mathcal{E}_3 = \langle \vec{e}_1, \vec{e}_2, \vec{e}_3 \rangle$. The subspace with the basis $B_1 = \langle \vec{e}_1 \rangle$ is the x-axis, the subspace with the basis $B_2 = \langle \vec{e}_2 \rangle$ is the y-axis, and the subspace with the basis $B_3 = \langle \vec{e}_3 \rangle$ is the z-axis. The fact that any member of $\mathbb{R}^3$ is expressible as a sum of vectors from these subspaces

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}$$

reflects the fact that $\mathcal{E}_3$ spans the space — this equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

has a solution for any $x, y, z \in \mathbb{R}$. And the fact that each such expression is unique reflects that fact that $\mathcal{E}_3$ is linearly independent, so any equation like the one above has a unique solution.

4.6 Example We don't have to take the basis vectors one at a time, we can conglomerate them into larger sequences. Consider again the space $\mathbb{R}^3$ and the vectors from the standard basis $\mathcal{E}_3$. The subspace with the basis $B_1 = \langle \vec{e}_1, \vec{e}_3 \rangle$ is the xz-plane. The subspace with the basis $B_2 = \langle \vec{e}_2 \rangle$ is the y-axis. As in the prior example, the fact that any member of the space is a sum of members of the two subspaces in one and only one way

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}$$

is a reflection of the fact that these vectors form a basis — this equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}) + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

has one and only one solution for any $x, y, z \in \mathbb{R}$.

4.7 Definition The *concatenation* of the sequences $B_1 = \langle \vec{\beta}_{1,1}, \dots, \vec{\beta}_{1,n_1} \rangle$, ..., $B_k = \langle \vec{\beta}_{k,1}, \dots, \vec{\beta}_{k,n_k} \rangle$ adjoins them into a single sequence.

$$B_1 {}^\frown B_2 {}^\frown \cdots {}^\frown B_k = \langle \vec{\beta}_{1,1}, \dots, \vec{\beta}_{1,n_1}, \vec{\beta}_{2,1}, \dots, \vec{\beta}_{k,n_k} \rangle$$

4.8 Lemma Let V be a vector space that is the sum of some of its subspaces $V = W_1 + \cdots + W_k$. Let B_1, ..., B_k be bases for these subspaces. The following are equivalent.

(1) The expression of any $\vec{v} \in V$ as a combination $\vec{v} = \vec{w}_1 + \cdots + \vec{w}_k$ with $\vec{w}_i \in W_i$ is unique.

(2) The concatenation $B_1 {}^\frown \cdots {}^\frown B_k$ is a basis for V.

(3) Among nonzero vectors from different W_i's every linear relationship is trivial.

Proof We will show that (1) $\implies$ (2), that (2) $\implies$ (3), and finally that (3) $\implies$ (1). For these arguments, observe that we can pass from a combination of $\vec{w}$'s to a combination of $\vec{\beta}$'s

$$\begin{aligned} d_1\vec{w}_1 + \cdots + d_k\vec{w}_k &= d_1(c_{1,1}\vec{\beta}_{1,1} + \cdots + c_{1,n_1}\vec{\beta}_{1,n_1}) \\ &\qquad + \cdots + d_k(c_{k,1}\vec{\beta}_{k,1} + \cdots + c_{k,n_k}\vec{\beta}_{k,n_k}) \\ &= d_1c_{1,1} \cdot \vec{\beta}_{1,1} + \cdots + d_kc_{k,n_k} \cdot \vec{\beta}_{k,n_k} \end{aligned} \qquad (*)$$

and vice versa (we can move from the bottom to the top by taking each d_i to be 1).

For (1) $\implies$ (2), assume that all decompositions are unique. We will show that $B_1 \frown \cdots \frown B_k$ spans the space and is linearly independent. It spans the space because the assumption that $V = W_1 + \cdots + W_k$ means that every $\vec{v}$ can be expressed as $\vec{v} = \vec{w}_1 + \cdots + \vec{w}_k$, which translates by equation $(*)$ to an expression of $\vec{v}$ as a linear combination of the $\vec{\beta}$'s from the concatenation. For linear independence, consider this linear relationship.

$$\vec{0} = c_{1,1}\vec{\beta}_{1,1} + \cdots + c_{k,n_k}\vec{\beta}_{k,n_k}$$

Regroup as in $(*)$ (that is, move from bottom to top) to get the decomposition $\vec{0} = \vec{w}_1 + \cdots + \vec{w}_k$. Because the zero vector obviously has the decomposition $\vec{0} = \vec{0} + \cdots + \vec{0}$, the assumption that decompositions are unique shows that each $\vec{w}_i$ is the zero vector. This means that $c_{i,1}\vec{\beta}_{i,1} + \cdots + c_{i,n_i}\vec{\beta}_{i,n_i} = \vec{0}$, and since each B_i is a basis we have the desired conclusion that all of the c's are zero.

For (2) $\implies$ (3) assume that the concatenation of the bases is a basis for the entire space. Consider a linear relationship among nonzero vectors from different W_i's. This might or might not involve a vector from W_1, or one from W_2, etc., so we write it $\vec{0} = \cdots + d_i\vec{w}_i + \cdots$. As in equation $(*)$ expand the vector.

$$\begin{aligned}
\vec{0} &= \cdots + d_i(c_{i,1}\vec{\beta}_{i,1} + \cdots + c_{i,n_i}\vec{\beta}_{i,n_i}) + \cdots \\
&= \cdots + d_ic_{i,1}\cdot\vec{\beta}_{i,1} + \cdots + d_ic_{i,n_i}\cdot\vec{\beta}_{i,n_i} + \cdots
\end{aligned}$$

The linear independence of $B_1 \frown \cdots \frown B_k$ gives that each coefficient $d_ic_{i,j}$ is zero. Since $\vec{w}_i$ is nonzero vector, at least one of the $c_{i,j}$'s is not zero, and thus d_i is zero. This holds for each d_i, and therefore the linear relationship is trivial.

Finally, for (3) $\implies$ (1), assume that among nonzero vectors from different W_i's any linear relationship is trivial. Consider two decompositions of a vector $\vec{v} = \cdots + \vec{w}_i + \cdots$ and $\vec{v} = \cdots + \vec{u}_j + \cdots$ where $\vec{w}_i \in W_i$ and $\vec{u}_j \in W_j$. Subtract one from the other to get a linear relationship, something like this (if there is no $\vec{u}_i$ or $\vec{w}_j$ then leave those out).

$$\vec{0} = \cdots + (\vec{w}_i - \vec{u}_i) + \cdots + (\vec{w}_j - \vec{u}_j) + \cdots$$

The case assumption that statement (3) holds implies that the terms each equal the zero vector $\vec{w}_i - \vec{u}_i = \vec{0}$. Hence decompositions are unique. QED

4.9 Definition A collection of subspaces $\{W_1, \ldots, W_k\}$ is *independent* if no nonzero vector from any W_i is a linear combination of vectors from the other subspaces $W_1, \ldots, W_{i-1}, W_{i+1}, \ldots, W_k$.

4.10 Definition A vector space V is the *direct sum* (or *internal direct sum*) of its subspaces $W_1, \ldots, W_k$ if $V = W_1 + W_2 + \cdots + W_k$ and the collection $\{W_1, \ldots, W_k\}$ is independent. We write $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$.

4.11 Example Our prototype works: $\mathbb{R}^3 = x\text{-axis} \oplus y\text{-axis} \oplus z\text{-axis}$.

4.12 Example The space of 2×2 matrices is this direct sum.

$$\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a, d \in \mathbb{R}\} \oplus \{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in \mathbb{R}\} \oplus \{\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \mid c \in \mathbb{R}\}$$

It is the direct sum of subspaces in many other ways as well; direct sum decompositions are not unique.

4.13 Corollary The dimension of a direct sum is the sum of the dimensions of its summands.

Proof In Lemma 4.8, the number of basis vectors in the concatenation equals the sum of the number of vectors in the sub-bases. QED

The special case of two subspaces is worth its own mention.

4.14 Definition When a vector space is the direct sum of two of its subspaces then they are *complements*.

4.15 Lemma A vector space V is the direct sum of two of its subspaces W_1 and W_2 if and only if it is the sum of the two $V = W_1 + W_2$ and their intersection is trivial $W_1 \cap W_2 = \{\vec{0}\,\}$.

Proof Suppose first that $V = W_1 \oplus W_2$. By definition, V is the sum of the two $V = W_1 + W_2$. To show that their intersection is trivial let $\vec{v}$ be a vector from $W_1 \cap W_2$ and consider the equation $\vec{v} = \vec{v}$. On that equation's left side is a member of W_1 and on the right is a member of W_2, which we can think of as a linear combination of members of W_2. But the two spaces are independent so the only way that a member of W_1 can be a linear combination of vectors from W_2 is if that member is the zero vector $\vec{v} = \vec{0}$.

For the other direction, suppose that V is the sum of two spaces with a trivial intersection. To show that V is a direct sum of the two we need only show that the spaces are independent — that no nonzero member of the first is expressible as a linear combination of members of the second, and vice versa. This holds because any relationship $\vec{w}_1 = c_1\vec{w}_{2,1} + \cdots + c_k\vec{w}_{2,k}$ (with $\vec{w}_1 \in W_1$ and $\vec{w}_{2,j} \in W_2$ for all j) shows that the vector on the left is also in W_2, since the right side is a combination of members of W_2. The intersection of these two spaces is trivial, so $\vec{w}_1 = \vec{0}$. The same argument works for any $\vec{w}_2$. QED

4.16 Example In $\mathbb{R}^2$ the x-axis and the y-axis are complements, that is, $\mathbb{R}^2 = x\text{-axis} \oplus y\text{-axis}$. A space can have more than one pair of complementary subspaces; another pair for $\mathbb{R}^2$ are the subspaces consisting of the lines $y = x$ and $y = 2x$.

4.17 Example In the space $F = \{a\cos\theta + b\sin\theta \mid a, b \in \mathbb{R}\}$, the subspaces $W_1 = \{a\cos\theta \mid a \in \mathbb{R}\}$ and $W_2 = \{b\sin\theta \mid b \in \mathbb{R}\}$ are complements. The prior example noted that a space can be decomposed into more than one pair of complements. In addition note that F can has more than one pair of complementary subspaces where the first in the pair is W_1 — another complement of W_1 is $W_3 = \{b\sin\theta + b\cos\theta \mid b \in \mathbb{R}\}$.

4.18 Example In $\mathbb{R}^3$, the xy-plane and the yz-planes are not complements, which is the point of the discussion following Example 4.4. One complement of the xy-plane is the z-axis.

Here is a natural question that arises from Lemma 4.15: for $k > 2$ is the simple sum $V = W_1 + \cdots + W_k$ also a direct sum if and only if the intersection of the subspaces is trivial?

4.19 Example If there are more than two subspaces then having a trivial intersection is not enough to guarantee unique decomposition (i.e., is not enough to ensure that the spaces are independent). In $\mathbb{R}^3$, let W_1 be the x-axis, let W_2 be the y-axis, and let W_3 be this.

$$W_3 = \{\begin{pmatrix} q \\ q \\ r \end{pmatrix} \mid q, r \in \mathbb{R}\}$$

The check that $\mathbb{R}^3 = W_1 + W_2 + W_3$ is easy. The intersection $W_1 \cap W_2 \cap W_3$ is trivial, but decompositions aren't unique.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y - x \\ 0 \end{pmatrix} + \begin{pmatrix} x \\ x \\ z \end{pmatrix} = \begin{pmatrix} x - y \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} y \\ y \\ z \end{pmatrix}$$

(This example also shows that this requirement is also not enough: that all pairwise intersections of the subspaces be trivial. See Exercise 30.)

In this subsection we have seen two ways to regard a space as built up from component parts. Both are useful; in particular we will use the direct sum definition at the end of the Chapter Five.

Exercises

✓ **4.20** Decide if $\mathbb{R}^2$ is the direct sum of each W_1 and W_2.

(a) $W_1 = \{\begin{pmatrix} x \\ 0 \end{pmatrix} \mid x \in \mathbb{R}\}$, $W_2 = \{\begin{pmatrix} x \\ x \end{pmatrix} \mid x \in \mathbb{R}\}$

(b) $W_1 = \{\begin{pmatrix} s \\ s \end{pmatrix} \mid s \in \mathbb{R}\}$, $W_2 = \{\begin{pmatrix} s \\ 1.1s \end{pmatrix} \mid s \in \mathbb{R}\}$

(c) $W_1 = \mathbb{R}^2$, $W_2 = \{\vec{0}\}$

(d) $W_1 = W_2 = \{\begin{pmatrix} t \\ t \end{pmatrix} \mid t \in \mathbb{R}\}$

(e) $W_1 = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} x \\ 0 \end{pmatrix} \mid x \in \mathbb{R}\}$, $W_2 = \{\begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y \end{pmatrix} \mid y \in \mathbb{R}\}$

✓ 4.21 Show that $\mathbb{R}^3$ is the direct sum of the xy-plane with each of these.

(a) the z-axis

(b) the line

$$\{\begin{pmatrix} z \\ z \\ z \end{pmatrix} \mid z \in \mathbb{R}\}$$

4.22 Is $\mathcal{P}_2$ the direct sum of $\{a + bx^2 \mid a, b \in \mathbb{R}\}$ and $\{cx \mid c \in \mathbb{R}\}$?

✓ 4.23 In $\mathcal{P}_n$, the *even* polynomials are the members of this set

$$\mathcal{E} = \{p \in \mathcal{P}_n \mid p(-x) = p(x) \text{ for all } x\}$$

and the *odd* polynomials are the members of this set.

$$\mathcal{O} = \{p \in \mathcal{P}_n \mid p(-x) = -p(x) \text{ for all } x\}$$

Show that these are complementary subspaces.

4.24 Which of these subspaces of $\mathbb{R}^3$

W_1: the x-axis, W_2: the y-axis, W_3: the z-axis,
W_4: the plane $x + y + z = 0$, W_5: the yz-plane

can be combined to

(a) sum to $\mathbb{R}^3$? (b) direct sum to $\mathbb{R}^3$?

✓ 4.25 Show that $\mathcal{P}_n = \{a_0 \mid a_0 \in \mathbb{R}\} \oplus \ldots \oplus \{a_n x^n \mid a_n \in \mathbb{R}\}$.

4.26 What is $W_1 + W_2$ if $W_1 \subseteq W_2$?

4.27 Does Example 4.5 generalize? That is, is this true or false: if a vector space V has a basis $\langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ then it is the direct sum of the spans of the one-dimensional subspaces $V = [\{\vec{\beta}_1\}] \oplus \ldots \oplus [\{\vec{\beta}_n\}]$?

4.28 Can $\mathbb{R}^4$ be decomposed as a direct sum in two different ways? Can $\mathbb{R}^1$?

4.29 This exercise makes the notation of writing '+' between sets more natural. Prove that, where $W_1, \ldots, W_k$ are subspaces of a vector space,

$$W_1 + \cdots + W_k = \{\vec{w}_1 + \vec{w}_2 + \cdots + \vec{w}_k \mid \vec{w}_1 \in W_1, \ldots, \vec{w}_k \in W_k\},$$

and so the sum of subspaces is the subspace of all sums.

4.30 (Refer to Example 4.19. This exercise shows that the requirement that pairwise intersections be trivial is genuinely stronger than the requirement only that the intersection of all of the subspaces be trivial.) Give a vector space and three subspaces W_1, W_2, and W_3 such that the space is the sum of the subspaces, the intersection of all three subspaces $W_1 \cap W_2 \cap W_3$ is trivial, but the pairwise intersections $W_1 \cap W_2$, $W_1 \cap W_3$, and $W_2 \cap W_3$ are nontrivial.

✓ 4.31 Prove that if $V = W_1 \oplus \ldots \oplus W_k$ then $W_i \cap W_j$ is trivial whenever $i \neq j$. This shows that the first half of the proof of Lemma 4.15 extends to the case of more than two subspaces. (Example 4.19 shows that this implication does not reverse; the other half does not extend.)

4.32 Recall that no linearly independent set contains the zero vector. Can an independent set of subspaces contain the trivial subspace?

✓ 4.33 Does every subspace have a complement?

✓ 4.34 Let W_1, W_2 be subspaces of a vector space.

(a) Assume that the set S_1 spans W_1, and that the set S_2 spans W_2. Can $S_1 \cup S_2$ span $W_1 + W_2$? Must it?

(b) Assume that S_1 is a linearly independent subset of W_1 and that S_2 is a linearly independent subset of W_2. Can $S_1 \cup S_2$ be a linearly independent subset of $W_1 + W_2$? Must it?

4.35 When we decompose a vector space as a direct sum, the dimensions of the subspaces add to the dimension of the space. The situation with a space that is given as the sum of its subspaces is not as simple. This exercise considers the two-subspace special case.

(a) For these subspaces of $\mathcal{M}_{2\times2}$ find $W_1 \cap W_2$, $\dim(W_1 \cap W_2)$, $W_1 + W_2$, and $\dim(W_1 + W_2)$.

$$W_1 = \{\begin{pmatrix}0 & 0\\ c & d\end{pmatrix} \mid c, d \in \mathbb{R}\} \qquad W_2 = \{\begin{pmatrix}0 & b\\ c & 0\end{pmatrix} \mid b, c \in \mathbb{R}\}$$

(b) Suppose that U and W are subspaces of a vector space. Suppose that the sequence $\langle\vec{\beta}_1, \ldots, \vec{\beta}_k\rangle$ is a basis for $U \cap W$. Finally, suppose that the prior sequence has been expanded to give a sequence $\langle\vec{\mu}_1, \ldots, \vec{\mu}_j, \vec{\beta}_1, \ldots, \vec{\beta}_k\rangle$ that is a basis for U, and a sequence $\langle\vec{\beta}_1, \ldots, \vec{\beta}_k, \vec{\omega}_1, \ldots, \vec{\omega}_p\rangle$ that is a basis for W. Prove that this sequence

$$\langle\vec{\mu}_1, \ldots, \vec{\mu}_j, \vec{\beta}_1, \ldots, \vec{\beta}_k, \vec{\omega}_1, \ldots, \vec{\omega}_p\rangle$$

is a basis for the sum $U + W$.

(c) Conclude that $\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$.

(d) Let W_1 and W_2 be eight-dimensional subspaces of a ten-dimensional space. List all values possible for $\dim(W_1 \cap W_2)$.

4.36 Let $V = W_1 \oplus \cdots \oplus W_k$ and for each index i suppose that S_i is a linearly independent subset of W_i. Prove that the union of the S_i's is linearly independent.

4.37 A matrix is *symmetric* if for each pair of indices i and j, the i, j entry equals the j, i entry. A matrix is *antisymmetric* if each i, j entry is the negative of the j, i entry.

(a) Give a symmetric 2×2 matrix and an antisymmetric 2×2 matrix. (*Remark.* For the second one, be careful about the entries on the diagonal.)

(b) What is the relationship between a square symmetric matrix and its transpose? Between a square antisymmetric matrix and its transpose?

(c) Show that $\mathcal{M}_{n\times n}$ is the direct sum of the space of symmetric matrices and the space of antisymmetric matrices.

4.38 Let W_1, W_2, W_3 be subspaces of a vector space. Prove that $(W_1 \cap W_2) + (W_1 \cap W_3) \subseteq W_1 \cap (W_2 + W_3)$. Does the inclusion reverse?

4.39 The example of the x-axis and the y-axis in $\mathbb{R}^2$ shows that $W_1 \oplus W_2 = V$ does not imply that $W_1 \cup W_2 = V$. Can $W_1 \oplus W_2 = V$ and $W_1 \cup W_2 = V$ happen?

✓ 4.40 Consider Corollary 4.13. Does it work both ways—that is, supposing that $V = W_1 + \cdots + W_k$, is $V = W_1 \oplus \cdots \oplus W_k$ if and only if $\dim(V) = \dim(W_1) + \cdots + \dim(W_k)$?

4.41 We know that if $V = W_1 \oplus W_2$ then there is a basis for V that splits into a basis for W_1 and a basis for W_2. Can we make the stronger statement that every basis for V splits into a basis for W_1 and a basis for W_2?

4.42 We can ask about the algebra of the '$+$' operation.

(a) Is it commutative; is $W_1 + W_2 = W_2 + W_1$?

(b) Is it associative; is $(W_1 + W_2) + W_3 = W_1 + (W_2 + W_3)$?

(c) Let W be a subspace of some vector space. Show that $W + W = W$.

(d) Must there be an identity element, a subspace I such that $I + W = W + I = W$ for all subspaces W?

(e) Does left-cancellation hold: if $W_1 + W_2 = W_1 + W_3$ then $W_2 = W_3$? Right cancellation?

4.43 Consider the algebraic properties of the direct sum operation.

(a) Does direct sum commute: does $V = W_1 \oplus W_2$ imply that $V = W_2 \oplus W_1$?

(b) Prove that direct sum is associative: $(W_1 \oplus W_2) \oplus W_3 = W_1 \oplus (W_2 \oplus W_3)$.

(c) Show that $\mathbb{R}^3$ is the direct sum of the three axes (the relevance here is that by the previous item, we needn't specify which two of the three axes are combined first).

(d) Does the direct sum operation left-cancel: does $W_1 \oplus W_2 = W_1 \oplus W_3$ imply $W_2 = W_3$? Does it right-cancel?

(e) There is an identity element with respect to this operation. Find it.

(f) Do some, or all, subspaces have inverses with respect to this operation: is there a subspace W of some vector space such that there is a subspace U with the property that $U \oplus W$ equals the identity element from the prior item?

Topic

Fields

Computations involving only integers or only rational numbers are much easier than those with real numbers. Could other algebraic structures, such as the integers or the rationals, work in the place of $\mathbb{R}$ in the definition of a vector space?

If we take "work" to mean that the results of this chapter remain true then there is a natural list of conditions that a structure (that is, number system) must have in order to work in the place of $\mathbb{R}$. A *field* is a set $\mathcal{F}$ with operations '$+$' and '$\cdot$' such that

(1) for any $a, b \in \mathcal{F}$ the result of $a + b$ is in $\mathcal{F}$, and $a + b = b + a$, and if $c \in \mathcal{F}$ then $a + (b + c) = (a + b) + c$

(2) for any $a, b \in \mathcal{F}$ the result of $a \cdot b$ is in $\mathcal{F}$, and $a \cdot b = b \cdot a$, and if $c \in \mathcal{F}$ then $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

(3) if $a, b, c \in \mathcal{F}$ then $a \cdot (b + c) = a \cdot b + a \cdot c$

(4) there is an element $0 \in \mathcal{F}$ such that if $a \in \mathcal{F}$ then $a + 0 = a$, and for each $a \in \mathcal{F}$ there is an element $-a \in \mathcal{F}$ such that $(-a) + a = 0$

(5) there is an element $1 \in \mathcal{F}$ such that if $a \in \mathcal{F}$ then $a \cdot 1 = a$, and for each element $a \neq 0$ of $\mathcal{F}$ there is an element $a^{-1} \in \mathcal{F}$ such that $a^{-1} \cdot a = 1$.

For example, the algebraic structure consisting of the set of real numbers along with its usual addition and multiplication operation is a field. Another field is the set of rational numbers with its usual addition and multiplication operations. An example of an algebraic structure that is not a field is the integers, because it fails the final condition.

Some examples are more surprising. The set $\mathbb{B} = \{0, 1\}$ under these operations:

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array} \qquad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}$$

is a field; see Exercise 4.

We could in this book develop Linear Algebra as the theory of vector spaces with scalars from an arbitrary field. In that case, almost all of the statements here would carry over by replacing '$\mathbb{R}$' with '$\mathcal{F}$', that is, by taking coefficients, vector entries, and matrix entries to be elements of $\mathcal{F}$ (the exceptions are statements involving distances or angles, which would need additional development). Here are some examples; each applies to a vector space V over a field $\mathcal{F}$.

* For any $\vec{v} \in V$ and $a \in \mathcal{F}$, (i) $0 \cdot \vec{v} = \vec{0}$, (ii) $-1 \cdot \vec{v} + \vec{v} = \vec{0}$, and (iii) $a \cdot \vec{0} = \vec{0}$.
* The span, the set of linear combinations, of a subset of V is a subspace of V.
* Any subset of a linearly independent set is also linearly independent.
* In a finite-dimensional vector space, any two bases have the same number of elements.

(Even statements that don't explicitly mention $\mathcal{F}$ use field properties in their proof.)

We will not develop vector spaces in this more general setting because the additional abstraction can be a distraction. The ideas we want to bring out already appear when we stick to the reals.

The exception is Chapter Five. There we must factor polynomials, so we will switch to considering vector spaces over the field of complex numbers.

Exercises

1 Check that the real numbers form a field.

2 Prove that these are fields.
(a) The rational numbers $\mathbb{Q}$ **(b)** The complex numbers $\mathbb{C}$

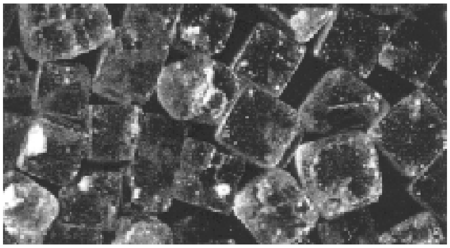
3 Give an example that shows that the integer number system is not a field.

4 Check that the set $\mathbb{B} = \{0, 1\}$ is a field under the operations listed above,

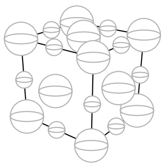
5 Give suitable operations to make the set $\{0, 1, 2\}$ a field.

Crystals

Everyone has noticed that table salt comes in little cubes.

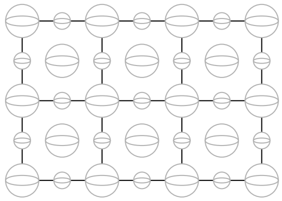


This orderly outside arises from an orderly inside — the way the atoms lie is also cubical, these cubes stack in neat rows and columns, and the salt faces tend to be just an outer layer of cubes. One cube of atoms is shown below. Salt is sodium chloride and the small spheres shown are sodium while the big ones are chloride. To simplify the view, it only shows the sodiums and chlorides on the front, top, and right.



The specks of salt that we see above have many repetitions of this fundamental unit. A solid, such as table salt, with a regular internal structure is a *crystal*.

We can restrict our attention to the front face. There we have a square repeated many times giving a lattice of atoms.

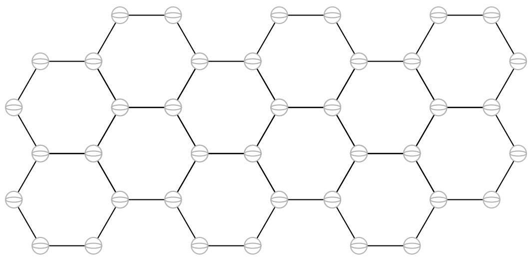


The distance along the sides of each square cell is about 3.34 Ångstroms (an Ångstrom is 10^{-10} meters). When we want to refer to atoms in the lattice that number is unwieldy, and so we take the square's side length as a unit. That is, we naturally adopt this basis.

$$\langle \begin{pmatrix} 3.34 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3.34 \end{pmatrix} \rangle$$

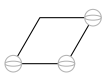
Now we can describe, say, the atom in the upper right of the lattice picture above as $3\vec{\beta}_1 + 2\vec{\beta}_2$, instead of 10.02 Ångstroms over and 6.68 up.

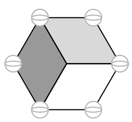
Another crystal from everyday experience is pencil lead. It is *graphite*, formed from carbon atoms arranged in this shape.



This is a single plane of graphite, called *graphene*. A piece of graphite consists of many of these planes, layered. The chemical bonds between the planes are much weaker than the bonds inside the planes, which explains why pencils write — the graphite can be sheared so that the planes slide off and are left on the paper.

We can get a convenient unit of length by decomposing the hexagonal ring into three regions that are rotations of this *unit cell*.



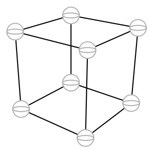


The vectors that form the sides of that unit cell make a convenient basis. The distance along the bottom and slant is 1.42 Ångstroms, so this

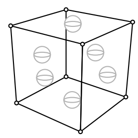
$$\langle \begin{pmatrix} 1.42 \\ 0 \end{pmatrix}, \begin{pmatrix} 1.23 \\ .71 \end{pmatrix} \rangle$$

is a good basis.

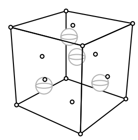
Another familiar crystal formed from carbon is diamond. Like table salt it is built from cubes but the structure inside each cube is more complicated. In addition to carbons at each corner,



there are carbons in the middle of each face.



(To show the new face carbons clearly, the corner carbons are reduced to dots.) There are also four more carbons inside the cube, two that are a quarter of the way up from the bottom and two that are a quarter of the way down from the top.



(As before, carbons shown earlier are reduced here to dots.) The distance along any edge of the cube is 2.18 Ångstroms. Thus, a natural basis for describing the locations of the carbons and the bonds between them, is this.

$$\langle \begin{pmatrix} 2.18 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2.18 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2.18 \end{pmatrix} \rangle$$

The examples here show that the structures of crystals is complicated enough to need some organized system to give the locations of the atoms and how they are chemically bound. One tool for that organization is a convenient basis. This application of bases is simple but it shows a science context where the idea arises naturally.

Exercises

1 How many fundamental regions are there in one face of a speck of salt? (With a ruler, we can estimate that face is a square that is 0.1 cm on a side.)

2 In the graphite picture, imagine that we are interested in a point 5.67 Ångstroms over and 3.14 Ångstroms up from the origin.

(a) Express that point in terms of the basis given for graphite.

(b) How many hexagonal shapes away is this point from the origin?

(c) Express that point in terms of a second basis, where the first basis vector is the same, but the second is perpendicular to the first (going up the plane) and of the same length.

3 Give the locations of the atoms in the diamond cube both in terms of the basis, and in Ångstroms.

4 This illustrates how we could compute the dimensions of a unit cell from the shape in which a substance crystallizes ([Ebbing], p. 462).

(a) Recall that there are 6.022×10^{23} atoms in a mole (this is Avogadro's number). From that, and the fact that platinum has a mass of 195.08 grams per mole, calculate the mass of each atom.

(b) Platinum crystallizes in a face-centered cubic lattice with atoms at each lattice point, that is, it looks like the middle picture given above for the diamond crystal. Find the number of platinum's per unit cell (hint: sum the fractions of platinum's that are inside of a single cell).

(c) From that, find the mass of a unit cell.

(d) Platinum crystal has a density of 21.45 grams per cubic centimeter. From this, and the mass of a unit cell, calculate the volume of a unit cell.

(e) Find the length of each edge.

(f) Describe a natural three-dimensional basis.

Topic

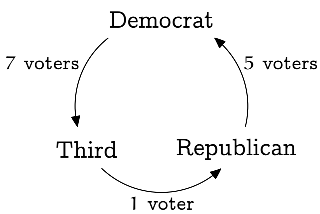
Voting Paradoxes

Imagine that a Political Science class studying the American presidential process holds a mock election. The 29 class members rank the Democratic Party, Republican Party, and Third Party nominees, from most preferred to least preferred (> means 'is preferred to').

preference order	*number with that preference*
Democrat > Republican > Third	5
Democrat > Third > Republican	4
Republican > Democrat > Third	2
Republican > Third > Democrat	8
Third > Democrat > Republican	8
Third > Republican > Democrat	2

What is the preference of the group as a whole?

Overall, the group prefers the Democrat to the Republican by five votes; seventeen voters ranked the Democrat above the Republican versus twelve the other way. And the group prefers the Republican to the Third's nominee, fifteen to fourteen. But, strangely enough, the group also prefers the Third to the Democrat, eighteen to eleven.

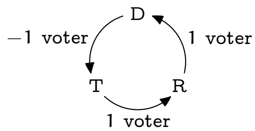


This is a *voting paradox*, specifically, a *majority cycle*.

Mathematicians study voting paradoxes in part because of their implications for practical politics. For instance, the instructor can manipulate this class into choosing the Democrat as the overall winner by first asking for a vote between the Republican and the Third, and then asking for a vote between the winner of that contest, who will be the Republican, and the Democrat. By similar manipulations the instructor can make any of the other two candidates come out as the winner. (We will stick to three-candidate elections but the same thing happens in larger elections.)

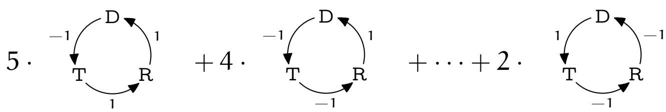
Mathematicians also study voting paradoxes simply because they are interesting. One interesting aspect is that the group's overall majority cycle occurs despite that each single voter's preference list is *rational*, in a straight-line order. That is, the majority cycle seems to arise in the aggregate without being present in the components of that aggregate, the preference lists. However we can use linear algebra to argue that a tendency toward cyclic preference is actually present in each voter's list and that it surfaces when there is more adding of the tendency than canceling.

For this, abbreviating the choices as D, R, and T, we can describe how a voter with preference order $D > R > T$ contributes to the above cycle.



(The negative sign is here because the arrow describes T as preferred to D, but this voter likes them the other way.) The descriptions for the other preference lists are in the table on page 153.

Now, to conduct the election we linearly combine these descriptions; for instance, the Political Science mock election



yields the circular group preference shown earlier.

Of course, taking linear combinations is linear algebra. The graphical cycle notation is suggestive but inconvenient so we use column vectors by starting at the D and taking the numbers from the cycle in counterclockwise order. Thus, we represent the mock election and a single $D > R > T$ vote in this way.

$$\begin{pmatrix}7\\1\\5\end{pmatrix} \quad\text{and}\quad \begin{pmatrix}-1\\1\\1\end{pmatrix}$$

We will decompose vote vectors into two parts, one cyclic and the other acyclic. For the first part, we say that a vector is *purely cyclic* if it is in this subspace of $\mathbb{R}^3$.

$$C = \{ \begin{pmatrix} k \\ k \\ k \end{pmatrix} \mid k \in \mathbb{R} \} = \{ k \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \mid k \in \mathbb{R} \}$$

For the second part, consider the set of vectors that are perpendicular to all of the vectors in C. Exercise 6 shows that this is a subspace

$$\begin{aligned} C^{\perp} &= \{ \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \mid \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \bullet \begin{pmatrix} k \\ k \\ k \end{pmatrix} = 0 \text{ for all } k \in \mathbb{R} \} \\ &= \{ \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \mid c_1 + c_2 + c_3 = 0 \} = \{ c_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \mid c_2, c_3 \in \mathbb{R} \} \end{aligned}$$

(read that aloud as "C perp"). So we are led to this basis for $\mathbb{R}^3$.

$$\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \rangle$$

We can represent votes with respect to this basis, and thereby decompose them into a cyclic part and an acyclic part. *(Note for readers who have covered the optional section in this chapter: that is, the space is the direct sum of* C *and* $C^{\perp}$*.)*

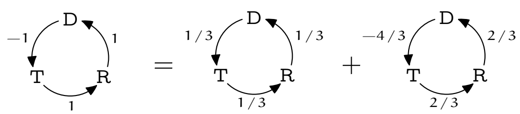
For example, consider the $D > R > T$ voter discussed above. We represent it with respect to the basis

$$\begin{array}{rcl} c_1 - c_2 - c_3 &=& -1 \\ c_1 + c_2 \phantom{{}-c_3} &=& 1 \\ c_1 \phantom{{}-c_2} + c_3 &=& 1 \end{array} \quad \xrightarrow[-\rho_1+\rho_3]{-\rho_1+\rho_2} \quad \xrightarrow{(-1/2)\rho_2+\rho_3} \quad \begin{array}{rcl} c_1 - c_2 - c_3 &=& -1 \\ 2c_2 + c_3 &=& 2 \\ (3/2)c_3 &=& 1 \end{array}$$

using the coordinates $c_1 = 1/3$, $c_2 = 2/3$, and $c_3 = 2/3$. Then

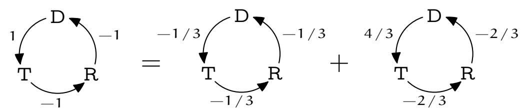
$$\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{3} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{2}{3} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \frac{2}{3} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix} + \begin{pmatrix} -4/3 \\ 2/3 \\ 2/3 \end{pmatrix}$$

gives the desired decomposition into a cyclic part and an acyclic part.



Thus we can see that this D > R > T voter's rational preference list does have a cyclic part.

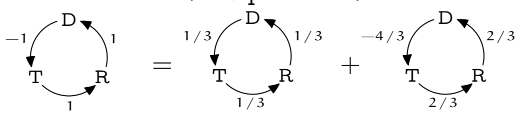
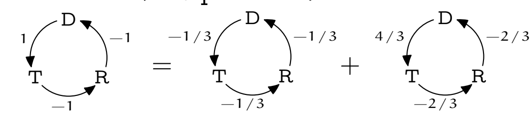
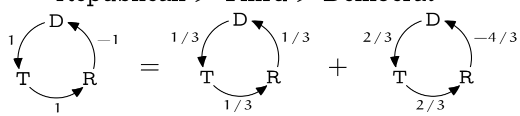
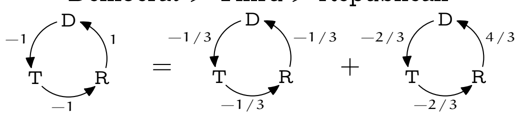
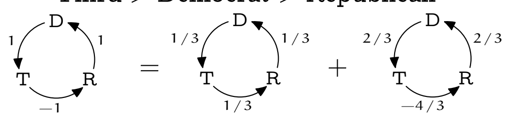
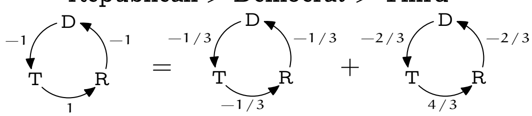
The T > R > D voter is opposite to the one just considered in that the '>' symbols are reversed. This voter's decomposition



shows that these opposite preferences have decompositions that are opposite. We say that the first voter has positive *spin* since the cycle part is with the direction that we have chosen for the arrows, while the second voter's spin is negative.

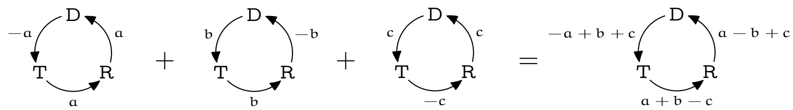
The fact that these opposite voters cancel each other is reflected in the fact that their vote vectors add to zero. This suggests an alternate way to tally an election. We could first cancel as many opposite preference lists as possible, and then determine the outcome by adding the remaining lists.

The table below contains the three pairs of opposite preference lists. For instance, the top line contains the voters discussed above.

positive spin	*negative spin*
Democrat > Republican > Third 	Third > Republican > Democrat 
Republican > Third > Democrat 	Democrat > Third > Republican 
Third > Democrat > Republican 	Republican > Democrat > Third 

If we conduct the election as just described then after the cancellation of as many opposite pairs of voters as possible there will remain three sets of preference lists: one set from the first row, one from the second row, and one from the third row. We will finish by proving that a voting paradox can happen only if the spins of these three sets are in the same direction. That is, for a voting paradox to occur the three remaining sets must all come from the left of the table or all come from the right (see Exercise 3). This shows that there is some connection between the majority cycle and the decomposition that we are using — a voting paradox can happen only when the tendencies toward cyclic preference reinforce each other.

For the proof, assume that we have canceled opposite preference orders and we are left with one set of preference lists from each of the three rows. Consider the sum of these three (here, the numbers a, b, and c could be positive, negative, or zero).



A voting paradox occurs when the three numbers on the right, $a - b + c$ and $a + b - c$ and $-a + b + c$, are all nonnegative or all nonpositive. On the left, at least two of the three numbers a and b and c are both nonnegative or both nonpositive. We can assume that they are a and b. That makes four cases: the cycle is nonnegative and a and b are nonnegative, the cycle is nonpositive and a and b are nonpositive, etc. We will do only the first case, since the second is similar and the other two are also easy.

So assume that the cycle is nonnegative and that a and b are nonnegative. The conditions $0 \leqslant a - b + c$ and $0 \leqslant -a + b + c$ add to give that $0 \leqslant 2c$, which implies that c is also nonnegative, as desired. That ends the proof.

This result says only that having all three spin in the same direction is a necessary condition for a majority cycle. It is not sufficient; see Exercise 4.

Voting theory and associated topics are the subject of current research. There are many intriguing results, notably the one produced by K Arrow [Arrow] who won the Nobel Prize in part for this work, showing that no voting system is entirely fair (for a reasonable definition of "fair"). Some good introductory articles are [Gardner, 1970], [Gardner, 1974], [Gardner, 1980], and [Neimi & Riker]. [Taylor] is a readable recent book. The long list of cases from recent American political history in [Poundstone] shows these paradoxes are routinely manipulated in practice.

This Topic is largely drawn from [Zwicker]. *(Author's Note: I would like to thank Professor Zwicker for his kind and illuminating discussions.)*

Exercises

1 Here is a reasonable way in which a voter could have a cyclic preference. Suppose that this voter ranks each candidate on each of three criteria.

(a) Draw up a table with the rows labeled 'Democrat', 'Republican', and 'Third', and the columns labeled 'character', 'experience', and 'policies'. Inside each column, rank some candidate as most preferred, rank another as in the middle, and rank the remaining one as least preferred.

(b) In this ranking, is the Democrat preferred to the Republican in (at least) two out of three criteria, or vice versa? Is the Republican preferred to the Third?

(c) Does the table that was just constructed have a cyclic preference order? If not, make one that does.

So it is possible for a voter to have a cyclic preference among candidates. The paradox described above, however, is that even if each voter has a straight-line preference list, a cyclic preference can still arise for the entire group.

2 Compute the values in the table of decompositions.

3 Do the cancellations of opposite preference orders for the Political Science class's mock election. Are all the remaining preferences from the left three rows of the table or from the right?

4 The necessary condition that is proved above — a voting paradox can happen only if all three preference lists remaining after cancellation have the same spin—is not also sufficient.

(a) Continuing the positive cycle case considered in the proof, use the two inequalities $0 \leqslant a - b + c$ and $0 \leqslant -a + b + c$ to show that $|a - b| \leqslant c$.

(b) Also show that $c \leqslant a + b$, and hence that $|a - b| \leqslant c \leqslant a + b$.

(c) Give an example of a vote where there is a majority cycle, and addition of one more voter with the same spin causes the cycle to go away.

(d) Can the opposite happen; can addition of one voter with a "wrong" spin cause a cycle to appear?

(e) Give a condition that is both necessary and sufficient to get a majority cycle.

5 A one-voter election cannot have a majority cycle because of the requirement that we've imposed that the voter's list must be rational.

(a) Show that a two-voter election may have a majority cycle. (We consider the group preference a majority cycle if all three group totals are nonnegative or if all three are nonpositive—that is, we allow some zero's in the group preference.)

(b) Show that for any number of voters greater than one, there is an election involving that many voters that results in a majority cycle.

6 Let U be a subspace of $\mathbb{R}^3$. Prove that the set $U^{\perp} = \{\vec{v} \mid \vec{v} \cdot \vec{u} = 0 \text{ for all } \vec{u} \in U\}$ of vectors that are perpendicular to each vector in U is also subspace of $\mathbb{R}^3$. Does this hold if U is not a subspace?

Topic

Dimensional Analysis

"You can't add apples and oranges," the old saying goes. It reflects our experience that in applications the quantities have units and keeping track of those units can help. Everyone has done calculations such as this one that use the units as a check.

$$60\,\frac{\text{sec}}{\text{min}}\cdot 60\,\frac{\text{min}}{\text{hr}}\cdot 24\,\frac{\text{hr}}{\text{day}}\cdot 365\,\frac{\text{day}}{\text{year}}=31\,536\,000\,\frac{\text{sec}}{\text{year}}$$

We can take the idea of including the units beyond bookkeeping. We can use units to draw conclusions about what relationships are possible among the physical quantities.

To start, consider the falling body equation $\text{distance}=16\cdot(\text{time})^2$. If the distance is in feet and the time is in seconds then this is a true statement. However it is not correct in other unit systems, such as meters and seconds, because 16 isn't the right constant in those systems. We can fix that by attaching units to the 16, making it a *dimensional constant*.

$$\text{dist}=16\,\frac{\text{ft}}{\text{sec}^2}\cdot(\text{time})^2$$

Now the equation holds also in the meter-second system because when we align the units (a foot is approximately 0.30 meters),

$$\text{distance in meters}=16\,\frac{0.30\,\text{m}}{\text{sec}^2}\cdot(\text{time in sec})^2=4.8\,\frac{\text{m}}{\text{sec}^2}\cdot(\text{time in sec})^2$$

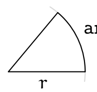
the constant gets adjusted. So in order to have equations that are correct across unit systems, we restrict our attention to those that use dimensional constants. Such an equation is *complete*.

Moving away from a particular unit system allows us to just measure quantities in combinations of some units of length L, mass M, and time T. These three are our *physical dimensions*. For instance, we could measure velocity in feet/second or fathoms/hour but at all events it involves a unit of length divided by a unit of time so the *dimensional formula* of velocity is L/T. Similarly, density's dimensional formula is M/L^3.

To write the dimensional formula we shall use negative exponents instead of fractions and we shall include the dimensions with a zero exponent. Thus we will write the dimensional formula of velocity as $L^1M^0T^{-1}$ and that of density as $L^{-3}M^1T^0$.

With that, "you can't add apples and oranges" becomes the advice to check that all of an equation's terms have the same dimensional formula. An example is this version of the falling body equation $d - gt^2 = 0$. The dimensional formula of the d term is $L^1M^0T^0$. For the other term, the dimensional formula of g is $L^1M^0T^{-2}$ (g is given above as $16\,\text{ft}/\text{sec}^2$) and the dimensional formula of t is $L^0M^0T^1$ so that of the entire gt^2 term is $L^1M^0T^{-2}(L^0M^0T^1)^2 = L^1M^0T^0$. Thus the two terms have the same dimensional formula. An equation with this property is *dimensionally homogeneous*.

Quantities with dimensional formula $L^0M^0T^0$ are *dimensionless*. For example, we measure an angle by taking the ratio of the subtended arc to the radius



which is the ratio of a length to a length $(L^1M^0T^0)(L^1M^0T^0)^{-1}$ and thus angles have the dimensional formula $L^0M^0T^0$.

The classic example of using the units for more than bookkeeping, using them to draw conclusions, considers the formula for the period of a pendulum.

$$p = \text{–some expression involving the length of the string, etc.–}$$

The period is in units of time $L^0M^0T^1$. So the quantities on the other side of the equation must have dimensional formulas that combine in such a way that their L's and M's cancel and only a single T remains. The table on page 158 has the quantities that an experienced investigator would consider possibly relevant to the period of a pendulum. The only dimensional formulas involving L are for the length of the string and the acceleration due to gravity. For the L's of these two to cancel when they appear in the equation they must be in ratio, e.g., as $(\ell/g)^2$, or as $\cos(\ell/g)$, or as $(\ell/g)^{-1}$. Therefore the period is a function of ℓ/g.

This is a remarkable result: with a pencil and paper analysis, before we ever took out the pendulum and made measurements, we have determined something about what makes up its period.

To do dimensional analysis systematically, we need two facts (arguments for these are in [Bridgman], Chapter II and IV). The first is that each equation relating physical quantities that we shall see involves a sum of terms, where each term has the form

$$m_1^{p_1} m_2^{p_2} \cdots m_k^{p_k}$$

for numbers m_1, ..., m_k that measure the quantities.

For the second fact, observe that an easy way to construct a dimensionally homogeneous expression is by taking a product of dimensionless quantities or by adding such dimensionless terms. Buckingham's Theorem states that any complete relationship among quantities with dimensional formulas can be algebraically manipulated into a form where there is some function f such that

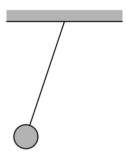
$$f(\Pi_1, \ldots, \Pi_n) = 0$$

for a complete set $\{\Pi_1, \ldots, \Pi_n\}$ of dimensionless products. (The first example below describes what makes a set of dimensionless products 'complete'.) We usually want to express one of the quantities, m_1 for instance, in terms of the others. For that we will assume that the above equality can be rewritten

$$m_1 = m_2^{-p_2} \cdots m_k^{-p_k} \cdot \hat{f}(\Pi_2, \ldots, \Pi_n)$$

where $\Pi_1 = m_1 m_2^{p_2} \cdots m_k^{p_k}$ is dimensionless and the products Π_2, ..., Π_n don't involve m_1 (as with f, here $\hat{f}$ is an arbitrary function, this time of $n-1$ arguments). Thus, to do dimensional analysis we should find which dimensionless products are possible.

For example, again consider the formula for a pendulum's period.



quantity	*dimensional formula*
period p	$L^0M^0T^1$
length of string ℓ	$L^1M^0T^0$
mass of bob m	$L^0M^1T^0$
acceleration due to gravity g	$L^1M^0T^{-2}$
arc of swing θ	$L^0M^0T^0$

By the first fact cited above, we expect the formula to have (possibly sums of terms of) the form $p^{p_1}\ell^{p_2}m^{p_3}g^{p_4}\theta^{p_5}$. To use the second fact, to find which combinations of the powers p_1, ..., p_5 yield dimensionless products, consider this equation.

$$(L^0M^0T^1)^{p_1}(L^1M^0T^0)^{p_2}(L^0M^1T^0)^{p_3}(L^1M^0T^{-2})^{p_4}(L^0M^0T^0)^{p_5} = L^0M^0T^0$$

It gives three conditions on the powers.

$$\begin{aligned} p_2 \qquad + p_4 &= 0 \\ p_3 \qquad\quad &= 0 \\ p_1 \qquad\quad - 2p_4 &= 0 \end{aligned}$$

Note that $p_3 = 0$ so the mass of the bob does not affect the period. Gaussian reduction and parametrization of that system gives this

$$\{\begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{pmatrix} = \begin{pmatrix} 1 \\ -1/2 \\ 0 \\ 1/2 \\ 0 \end{pmatrix} p_1 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} p_5 \mid p_1, p_5 \in \mathbb{R}\}$$

(we've taken p_1 as one of the parameters in order to express the period in terms of the other quantities).

The set of dimensionless products contains all terms $p^{p_1}\ell^{p_2}m^{p_3}a^{p_4}\theta^{p_5}$ subject to the conditions above. This set forms a vector space under the '$+$' operation of multiplying two such products and the '$\cdot$' operation of raising such a product to the power of the scalar (see Exercise 5). The term 'complete set of dimensionless products' in Buckingham's Theorem means a basis for this vector space.

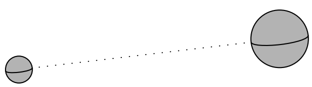
We can get a basis by first taking $p_1 = 1$, $p_5 = 0$, and then taking $p_1 = 0$, $p_5 = 1$. The associated dimensionless products are $\Pi_1 = p\ell^{-1/2}g^{1/2}$ and $\Pi_2 = \theta$. Because the set $\{\Pi_1, \Pi_2\}$ is complete, Buckingham's Theorem says that

$$p = \ell^{1/2}g^{-1/2} \cdot \hat{f}(\theta) = \sqrt{\ell/g} \cdot \hat{f}(\theta)$$

where $\hat{f}$ is a function that we cannot determine from this analysis (a first year physics text will show by other means that for small angles it is approximately the constant function $\hat{f}(\theta) = 2\pi$).

Thus, analysis of the relationships that are possible between the quantities with the given dimensional formulas has given us a fair amount of information: a pendulum's period does not depend on the mass of the bob, and it rises with the square root of the length of the string.

For the next example we try to determine the period of revolution of two bodies in space orbiting each other under mutual gravitational attraction. An experienced investigator could expect that these are the relevant quantities.



quantity	*dimensional formula*
period p	$L^0M^0T^1$
mean separation r	$L^1M^0T^0$
first mass m_1	$L^0M^1T^0$
second mass m_2	$L^0M^1T^0$
gravitational constant G	$L^3M^{-1}T^{-2}$

To get the complete set of dimensionless products we consider the equation

$$(L^0M^0T^1)^{p_1}(L^1M^0T^0)^{p_2}(L^0M^1T^0)^{p_3}(L^0M^1T^0)^{p_4}(L^3M^{-1}T^{-2})^{p_5} = L^0M^0T^0$$

which results in a system

$$\begin{array}{rcrcrcrcrcl} & & p_2 & & & & & + & 3p_5 & = & 0 \\ & & & & p_3 & + & p_4 & - & p_5 & = & 0 \\ p_1 & & & & & & & - & 2p_5 & = & 0 \end{array}$$

with this solution.

$$\{\begin{pmatrix} 1 \\ -3/2 \\ 1/2 \\ 0 \\ 1/2 \end{pmatrix} p_1 + \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} p_4 \mid p_1, p_4 \in \mathbb{R}\}$$

As earlier, the set of dimensionless products of these quantities forms a vector space and we want to produce a basis for that space, a 'complete' set of dimensionless products. One such set, gotten from setting $p_1 = 1$ and $p_4 = 0$ and also setting $p_1 = 0$ and $p_4 = 1$ is $\{\Pi_1 = pr^{-3/2}m_1^{1/2}G^{1/2},\ \Pi_2 = m_1^{-1}m_2\}$. With that, Buckingham's Theorem says that any complete relationship among these quantities is stateable this form.

$$p = r^{3/2}m_1^{-1/2}G^{-1/2} \cdot \hat{f}(m_1^{-1}m_2) = \frac{r^{3/2}}{\sqrt{Gm_1}} \cdot \hat{f}(m_2/m_1)$$

Remark. An important application of the prior formula is when m_1 is the mass of the sun and m_2 is the mass of a planet. Because m_1 is very much greater than m_2, the argument to $\hat{f}$ is approximately 0, and we can wonder whether this part of the formula remains approximately constant as m_2 varies. One way to see that it does is this. The sun is so much larger than the planet that the mutual rotation is approximately about the sun's center. If we vary the planet's mass m_2 by a factor of x (e.g., Venus's mass is $x = 0.815$ times Earth's mass), then the force of attraction is multiplied by x, and x times the force acting on x times the mass gives, since $F = ma$, the same acceleration, about the same center (approximately). Hence, the orbit will be the same and so its period will be the same, and thus the right side of the above equation also remains unchanged (approximately). Therefore, $\hat{f}(m_2/m_1)$ is approximately constant as m_2 varies. This is Kepler's Third Law: the square of the period of a planet is proportional to the cube of the mean radius of its orbit about the sun.

The final example was one of the first explicit applications of dimensional analysis. Lord Raleigh considered the speed of a wave in deep water and suggested these as the relevant quantities.

quantity	*dimensional formula*
velocity of the wave v	$L^1M^0T^{-1}$
density of the water d	$L^{-3}M^1T^0$
acceleration due to gravity g	$L^1M^0T^{-2}$
wavelength λ	$L^1M^0T^0$

The equation

$$(L^1M^0T^{-1})^{p_1}(L^{-3}M^1T^0)^{p_2}(L^1M^0T^{-2})^{p_3}(L^1M^0T^0)^{p_4} = L^0M^0T^0$$

gives this system

$$\begin{aligned} p_1 - 3p_2 + \ \ p_3 + p_4 &= 0 \\ p_2 \qquad\qquad &= 0 \\ -p_1 \qquad - 2p_3 \qquad &= 0 \end{aligned}$$

with this solution space.

$$\{\begin{pmatrix} 1 \\ 0 \\ -1/2 \\ -1/2 \end{pmatrix} p_1 \mid p_1 \in \mathbb{R}\}$$

There is one dimensionless product, $\Pi_1 = vg^{-1/2}\lambda^{-1/2}$, and so v is $\sqrt{\lambda g}$ times a constant; $\hat{f}$ is constant since it is a function of no arguments. The quantity d is not involved in the relationship.

The three examples above show that dimensional analysis can bring us far toward expressing the relationship among the quantities. For further reading, the classic reference is [Bridgman]—this brief book is delightful. Another source is [Giordano, Wells, Wilde]. A description of dimensional analysis's place in modeling is in [Giordano, Jaye, Weir].

Exercises

1 [de Mestre] Consider a projectile, launched with initial velocity v_0, at an angle θ. To study its motion we may guess that these are the relevant quantities.

quantity	*dimensional formula*
horizontal position x	$L^1M^0T^0$
vertical position y	$L^1M^0T^0$
initial speed v_0	$L^1M^0T^{-1}$
angle of launch θ	$L^0M^0T^0$
acceleration due to gravity g	$L^1M^0T^{-2}$
time t	$L^0M^0T^1$

(a) Show that $\{gt/v_0, gx/v_0^2, gy/v_0^2, \theta\}$ is a complete set of dimensionless products. (*Hint.* One way to go is to find the appropriate free variables in the linear system that arises but there is a shortcut that uses the properties of a basis.)

(b) These two equations of motion for projectiles are familiar: $x = v_0 \cos(\theta)t$ and $y = v_0 \sin(\theta)t - (g/2)t^2$. Manipulate each to rewrite it as a relationship among the dimensionless products of the prior item.

2 [Einstein] conjectured that the infrared characteristic frequencies of a solid might be determined by the same forces between atoms as determine the solid's ordinary elastic behavior. The relevant quantities are these.

quantity	*dimensional formula*
characteristic frequency ν	$L^0M^0T^{-1}$
compressibility k	$L^1M^{-1}T^2$
number of atoms per cubic cm N	$L^{-3}M^0T^0$
mass of an atom m	$L^0M^1T^0$

Show that there is one dimensionless product. Conclude that, in any complete relationship among quantities with these dimensional formulas, k is a constant times $\nu^{-2}N^{-1/3}m^{-1}$. This conclusion played an important role in the early study of quantum phenomena.

3 [Giordano, Wells, Wilde] The torque produced by an engine has dimensional formula $L^2M^1T^{-2}$. We may first guess that it depends on the engine's rotation rate (with dimensional formula $L^0M^0T^{-1}$), and the volume of air displaced (with dimensional formula $L^3M^0T^0$).

(a) Try to find a complete set of dimensionless products. What goes wrong?

(b) Adjust the guess by adding the density of the air (with dimensional formula $L^{-3}M^1T^0$). Now find a complete set of dimensionless products.

4 [Tilley] Dominoes falling make a wave. We may conjecture that the wave speed v depends on the spacing d between the dominoes, the height h of each domino, and the acceleration due to gravity g.

(a) Find the dimensional formula for each of the four quantities.

(b) Show that $\{\Pi_1 = h/d, \Pi_2 = dg/v^2\}$ is a complete set of dimensionless products.

(c) Show that if h/d is fixed then the propagation speed is proportional to the square root of d.

5 Prove that the dimensionless products form a vector space under the $\vec{+}$ operation of multiplying two such products and the $\vec{\cdot}$ operation of raising such the product to the power of the scalar. (The vector arrows are a precaution against confusion.) That is, prove that, for any particular homogeneous system, this set of products of powers of $m_1, \ldots, m_k$

$$\{m_1^{p_1} \ldots m_k^{p_k} \mid p_1, \ldots, p_k \text{ satisfy the system}\}$$

is a vector space under:

$$m_1^{p_1} \ldots m_k^{p_k} \vec{+} m_1^{q_1} \ldots m_k^{q_k} = m_1^{p_1+q_1} \ldots m_k^{p_k+q_k}$$

and

$$r \vec{\cdot} (m_1^{p_1} \ldots m_k^{p_k}) = m_1^{rp_1} \ldots m_k^{rp_k}$$

(assume that all variables represent real numbers).

6 The advice about apples and oranges is not right. Consider the familiar equations for a circle $C = 2\pi r$ and $A = \pi r^2$.

(a) Check that C and A have different dimensional formulas.

(b) Produce an equation that is not dimensionally homogeneous (i.e., it adds apples and oranges) but is nonetheless true of any circle.

(c) The prior item asks for an equation that is complete but not dimensionally homogeneous. Produce an equation that is dimensionally homogeneous but not complete.

(Just because the old saying isn't strictly right, doesn't keep it from being a useful strategy. Dimensional homogeneity is often used to check the plausibility of equations used in models. For an argument that any complete equation can easily be made dimensionally homogeneous, see [Bridgman], Chapter I, especially page 15.)